# Written Test

Dynamics of Structures

July 20, 2018

## 1 2 DoF system — Support Motion

**Eigenvalues** The equation of free vibration is

$$\left(\frac{EJ}{L^3}\begin{bmatrix}3 & -3\\ -3 & 6\end{bmatrix} - m\omega^2\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}\right)\boldsymbol{\psi}\,\sin\omega t = \mathbf{0},$$

that admits non trivial solutions when

$$\det\left(\frac{EJ}{L^3}\begin{bmatrix}3 & -3\\ -3 & 6\end{bmatrix} - m\omega^2\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}\right) = 0.$$

With $\omega^2 = \Lambda\omega_0^2$ we can write equivalently

$$(3-2\Lambda)(6-5\Lambda) = 9 \Leftrightarrow 9 - 27\Lambda + 10\Lambda^2 = 0 \Leftrightarrow \Lambda^2 - 2\times 1.35\Lambda + 0.9 = 0$$

that offers the solutions

$$\Lambda_1 = 1.35 - \sqrt{1.35^2 - 0.9} = 0.390, \quad \Lambda_2 = 1.35 + \sqrt{1.35^2 - 0.9} = 2.310.$$

**Eigenvectors** The first equation of the e.o.f.v. gives

$$\lambda_{2i} = \frac{3-2\Lambda_i}{3}\lambda_{1i}$$

and we may conveniently use (because we are going to compute $x_1$) $\psi_{11} = \psi_{12} = 1$ that, substituted in the previous equation, give

$$\boldsymbol{\Psi} = \begin{bmatrix}1 & 1\\ +0.740 & -0.540\end{bmatrix}.$$

It's easy to see that

$$\boldsymbol{\Psi}^{-1} = \frac{1}{1.280}\begin{bmatrix}0.540 & +1\\ 0.740 & -1\end{bmatrix}.$$

**Sketches of $\boldsymbol{u_E}$, $\dot{\boldsymbol{u}}_{\boldsymbol{E}}$, $\ddot{\boldsymbol{u}}_{\boldsymbol{E}}$** I will omit the trivial sketches of $u_E$ and its time derivatives, I'd just like to mention that $\dot{u}_E = \text{const.} = \omega_0\Delta$.

**Modal equations of motion** Because the acceleration of the support is always equal to zero (except for two *infinite impulses* of infinitesimal duration, that provide a discrete change of the velocity) there are NO apparent forces and our modal equation of motion are simply

$$\ddot{q}_1 + 0.390\,\omega_0^2 q_1 = 0,$$
$$\ddot{q}_2 + 2.310\,\omega_0^2 q_2 = 0.$$

While we are at it, we remark that the response is simply the homogeneous integral, with $\lambda_i = \sqrt{\Lambda_i}$

$$q_i(t) = E_i \sin\lambda_i\omega_0 t + B_i \cos\lambda_i\omega_0 t.$$

The response, clearly, depends on the initial conditions...

**Initial conditions** We write, again

$$\boldsymbol{x}_{\text{tot}} = \boldsymbol{x}_{\text{stat}} + \boldsymbol{x},$$
$$\dot{\boldsymbol{x}}_{\text{tot}} = \dot{\boldsymbol{x}}_{\text{stat}} + \dot{\boldsymbol{x}}.$$

Because $\boldsymbol{x}_{\text{stat}} = \boldsymbol{E}\,u_E$ we can write

$$\boldsymbol{x}_0 = \boldsymbol{x}_{\text{tot},0} - \boldsymbol{E}u_{E,0} = \boldsymbol{x}_{\text{tot},0}, \qquad \text{(because } u_{E,0} = 0\text{)}$$
$$\dot{\boldsymbol{x}}_0 = \dot{\boldsymbol{x}}_{\text{tot},0} - \boldsymbol{E}\dot{u}_{E,0} = \dot{\boldsymbol{x}}_{\text{tot},0} - \boldsymbol{E}\,\omega_0\Delta.$$

In terms of total displacements, both the initial total displacements and the initial total velocities are equal to zero, so that finally we have

$$\boldsymbol{x}_0 = \boldsymbol{0},$$
$$\dot{\boldsymbol{x}}_0 = -\boldsymbol{E}\,\omega_0\Delta.$$

Having said that $\boldsymbol{E} = \begin{bmatrix}1 & 1/2\end{bmatrix}^T$ we can now write the initial conditions in terms of modal coordinates:

$$\begin{aligned}\boldsymbol{q}_0 &= \boldsymbol{0},\\ \dot{\boldsymbol{q}}_0 &= -\Psi^{-1}\boldsymbol{E}\,\omega_0\Delta,\\ &= -\frac{1}{1.280}\begin{bmatrix}0.540 & +1\\ 0.740 & -1\end{bmatrix}\begin{bmatrix}1\\ 1/2\end{bmatrix}\omega_0\Delta,\\ &= -\frac{1}{16}\begin{bmatrix}13\\ 3\end{bmatrix}\omega_0\Delta.\end{aligned}$$

and

$$q_1 = -\frac{13}{16\lambda_1}\sin(\lambda_1\omega_0 t)\Delta,$$
$$q_2 = -\frac{3}{16\lambda_2}\sin(\lambda_2\omega_0 t)\Delta.$$

**Mass displacement at $t_1$** Because $\omega_0 t_1 = 1$ and $\psi_{11} = \psi_{12} = 1$,

$$x_1(t_1) = -\left({}^{13}\!/\!_{\lambda_1}\sin\lambda_1 + {}^{3}\!/\!_{\lambda_2}\sin\lambda_2\right)\frac{\Delta}{16}.$$

**What happens for $t > t_1$?** At $t = t_1$ the system will experience the opposite of what happened at $t = 0$, with a negative infinite acceleration impulse that takes back the support velocity to 0.

Mathematically, $q_i = A_i\big(\sin(\lambda_i\omega_0 t) - \sin(\lambda_i\omega_0(t - t_1))\big)$.

## 2 Free Vibrations

- The first two boundaries conditions are

  $$\phi(0)=0, \qquad M(0)=-EJ\phi''(0)=0$$

  and the coefficients of the cosine and of the hyperbolic cosine are hence zero and the general integral can be simplified:

  $$\phi(x)=A\sin\beta x+B\sinh\beta x.$$

  By $M(L)=-EJ\phi''(L)=0$ we have

  $$(\sin\beta L)\,A+(-\sinh\beta L)\,B=0.$$

  The vertical equilibrium condition at $x=L$, when $V=-EJ\phi'''$ is a clock-wise shear, is $V+k\phi(L)=0$ or, rearranging, $\phi(L)^{k}/_{EJ}=\phi(L)^{6}/_{L^3}=\phi'''(L)$. Expanding,

  $$(6\sin\beta L)\,A+(6\sinh\beta L)\,B=(-\beta^3L^3\cos\beta L)\,A+(\beta^3L^3\cosh\beta L)\,B.$$

  Representing our conditions as a linear system,

  $$\begin{bmatrix} +\sin\beta L & -\sinh\beta L \\ +6\sin\beta L+\beta^3L^3\cos\beta L & +6\sinh\beta L-\beta^3L^3\cosh\beta L \end{bmatrix}\begin{Bmatrix}A\\B\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

  we recognize the possibility of a non trivial solution when

  $$+\sin\beta L(+6\sinh\beta L-\beta^3L^3\cosh\beta)=-\sinh\beta L(+6\sin\beta L+\beta^3L^3\cos\beta L)$$

  Numerically, $\beta_1 L\simeq 2.0015$ and $\omega_1^2=(\beta_1L)^4\,\omega_0^2\simeq 16.05\,\omega_0^2$

- For a 1 DoF, we compute the flexibility using the PVD

  $$1\cdot\delta=\int_0^{\frac{2L}{3}}\frac{x^2}{9EJ}\,dx+\int_0^{\frac{L}{3}}\frac{4x^2}{9EJ}\,dx+\frac{2}{3}\cdot\frac{2/3}{k}=\frac{22}{243}\frac{L^3}{EJ}.$$

  The stiffness is $k^*=\frac{243}{22}\frac{EJ}{L^3}$, the mass is $m^*=\frac{2}{3}\bar{m}L$ and the first frequency is $\omega^2=\frac{729}{44}\omega_0^2\simeq 16.57\omega_0^2$.

- For the 2 DoF system, the mass matrix is

  $$\boldsymbol{M}=\bar{m}L\begin{bmatrix}1/2 & 0\\ 0 & 1/4\end{bmatrix},$$

  the stiffness matrix, that can be found computing the flexibilities with the PVD and inverting the flexibility matrix is

  $$\boldsymbol{K}=\frac{EJ}{L^3}\begin{bmatrix}+48 & -24\\ -24 & +18\end{bmatrix}$$

  and solving the eigenvalue problem gives $\omega^2=15.065\omega_0^2$.

  It is worth noting that this estimate is smaller than the "true" value of the first frequency: a model that uses lumped masses gives no guarantee with respect to convergence from above.

The Rayleigh quotient uses $v(x,t) = Z_0\psi(x)\sin\omega t$ and $\dot v(x,t) = \omega Z_0\psi(x)\cos\omega t$, and for our problem

$$V_{\max} = \frac{1}{2}Z_0^2\int_0^L EJ\psi''^2\,dx + \frac{1}{2}Z_0^2\psi(L)^2k,\quad T_{\max} = \frac{1}{2}\omega_0^2 Z_0^2\int_0^L \bar m\psi^2\,dx$$

and

$$\omega^2 = \frac{EJ\int_0^L \psi''^2\,dx + k\psi(L)^2}{\int_0^L \bar m\psi^2\,dx}$$

- With $\psi_a = \sin{}^{\pi x}/_L$ it is

$$\omega^2 = \frac{\frac{\pi^4}{L^4}EJ\int_0^L \sin^2{}^{\pi x}/_L\,dx + k\cdot 0^2}{\bar m\int_0^L \sin^2{}^{\pi x}/_L\,dx} = \frac{\frac{\pi^4}{2}\frac{EJ}{L^3}}{{}^{\bar m L}/_2} = \pi^4\omega_0^2.$$

- With $\psi_b = {}^x/_L$ it is

$$\omega^2 = \frac{EJ\int_0^L 0^2\,dx + k\cdot 1^2}{\bar m\int_0^L {}^{x^2}/_{L^2}\,dx} = \frac{\frac{6EJ}{L^3}}{{}^{\bar m L}/_3} = 18\omega_0^2.$$

- With $\psi = a\psi_a + b\psi_b$

$$\omega^2 = \frac{a^2\frac{\pi^4}{2}\frac{EJ}{L^3} + b^2\frac{6EJ}{L^3}}{({}^1/_2\,a^2 + {}^2/_\pi\,ab + {}^1/_3\,b^2)\bar m L} = \frac{a^2\frac{\pi^4}{2} + 6b^2}{({}^1/_2\,a^2 + {}^2/_\pi\,ab + {}^1/_3\,b^2)}\,\omega_0^2.$$

  Someone will recognize the ratio of two quadratic forms and understand that the minimum $\omega^2$ can be found by solving the eigenvalue problem formulated in Ritz coordinates:

$$\left(\begin{bmatrix}\frac{\pi^4}{2} & 0\\ 0 & 6\end{bmatrix} - \omega^2\begin{bmatrix}\frac{1}{2} & \frac{1}{\pi}\\ \frac{1}{\pi} & \frac{1}{3}\end{bmatrix}\right)\boldsymbol{z} = \boldsymbol{0}.$$

  Eventually one can find that $\omega^2 = 16.070\omega_0^2$. The corresponding shape function is depicted below.