Continuous System

Imports, stuff

```
from sympy import *
init_printing(use_latex=True)
from IPython.display import Latex
%matplotlib inline
```

Symbols

```
x, w2 = symbols('x omega^2')
L, m, EJ = symbols('L m EJ', positive = True)
A, B, C, D, ld, LD = symbols('A B C D lambda Lambda')
f, ϕ = symbols('f phi')
```

Supported mass and stiffness of support

```
mass_coeff = 8
stiff_coeff = 24
k = stiff_coeff*EJ/L**3
M = mass_coeff*m*L
```

General solution and its derivatives

```
f0 = A*cos(ld*x) + B*sin(ld*x) + C*cosh(ld*x) + D*sinh(ld*x)
f1 = f0.diff(x)
f2 = f1.diff(x)
f3 = f2.diff(x)
display(Eq(ϕ,f0))
```

$$\phi = A\cos(\lambda x) + B\sin(\lambda x) + C\cosh(\lambda x) + D\sinh(\lambda x)$$

Left boundary conditions

The eigenfunction and its second derivative must be zero when `0` is substituted for `x`, we solve for `A` and `C` and put the solution in the variable `AC`. We substitute our solution in the eigenfunctions and all of its derivatives.

```
AC = solve((f0.subs(x,0), f2.subs(x,0)), A, C, dict=True)
f0, f1, f2, f3 = [f.subs(AC[0]) for f in (f0, f1, f2, f3)]
display(Eq(ϕ, f0))
```

$$\phi = B\sin(\lambda x) + D\sinh(\lambda x)$$

First, simpler boundary condition at the right end, $x = L$.

The second derivative must be equal to zero, so we solve and substitute, also substitute λL with Λ

```
D = solve(f2.subs(x, L), D, dict=True)
f0, f1, f2, f3 = [f.subs(D[0]).subs(L,LD/ld) for f in (f0, f1, f2, f3)]
display(Latex('With $\\Lambda = \\lambda\\,L$ it is'))
display(Eq(ϕ, f0.simplify()))
```

With $\Lambda = \lambda\,L$ it is

$$\phi = B\left(\frac{\sin(\Lambda)\sinh(\lambda x)}{\sinh(\Lambda)} + \sin(\lambda x)\right)$$

Last boundary conditions, equation of wave numbers

The last equation is an equation of equilibrium

$$V(t) + k\,v(t) + M\,\ddot v(t) = 0$$

(all the forces are directed upwards).

With $v(t) = \phi(x)\,\sin\omega t$, the shear is $V = -EJ\,v''' = -EJ\,\phi'''(x)\sin\omega t$ and the inertial force is $M\,\ddot v = -M\,\phi\,\omega^2\sin\omega t$ that can be rewritten taking into account that $\omega^2 = \lambda^4 EJ/m$: $M\,\ddot v = -M/m\,EJ\,\lambda^4\phi\sin\omega t$.

Let's write the expanded equation, collecting all the terms that are no Λ:

```
eq = (f0*k - f0*M*ld**4*EJ/m - EJ*f3).subs(x, L).subs(L, LD/ld)
display(Eq(eq.expand().collect(B).collect(ld).collect(EJ), 0))
```

$$BEJ\lambda^3\left(-16\Lambda\sin(\Lambda) - \frac{\sin(\Lambda)\cosh(\Lambda)}{\sinh(\Lambda)} + \cos(\Lambda) + \frac{48\sin(\Lambda)}{\Lambda^3}\right) = 0$$

We have a non trivial solution when the term in brackets is equal to zero, to have the bracketed term we must divide both members by $B\,EJ\,\lambda^3$

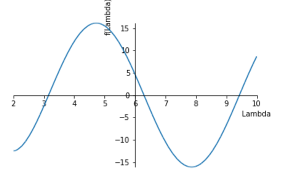
```
eq = (eq/EJ/ld**3/B).expand()
display(Eq(eq,0))
```

$$-16\Lambda\sin(\Lambda) - \frac{\sin(\Lambda)\cosh(\Lambda)}{\sinh(\Lambda)} + \cos(\Lambda) + \frac{48\sin(\Lambda)}{\Lambda^3} = 0$$

The behavior near $\Lambda = 0$ is led by the last term that goes like $48/\Lambda^2$, so to have a nice plot we multiply everything by Λ^2

```
display(Eq(symbols('f'), (eq*LD**2).expand()))
plot(eq*LD**2, (LD, 0, 2));
```

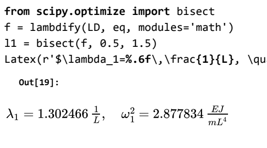
$$f = -16\Lambda^3\sin(\Lambda) - \frac{\Lambda^2\sin(\Lambda)\cosh(\Lambda)}{\sinh(\Lambda)} + \Lambda^2\cos(\Lambda) + \frac{48\sin(\Lambda)}{\Lambda}$$



and see that there is a root between 1.25 and 1.5. If we were interested in upper roots, we can observe that all the terms in the LHS of our determinantal equations are bounded for increasing Λ except for the first one, that grows linearly, so to investigate the other roots we may divide the equation by Λ to remove that trend...

```
display(Eq(symbols('f'), (eq/LD).expand()))
plot(eq/LD, (LD, 2, 10));
```

$$f = -16\sin(\Lambda) - \frac{\sin(\Lambda)\cosh(\Lambda)}{\Lambda\sinh(\Lambda)} + \frac{\cos(\Lambda)}{\Lambda} + \frac{48\sin(\Lambda)}{\Lambda^4}$$



All the RHS terms except the first have Λ in the denominator and are bounded, so the asymptotic behaviour is controlled by $\Lambda_{n+1} = n\pi$.

```
from scipy.optimize import bisect
f = lambdify(LD, eq, modules='math')
l1 = bisect(f, 0.5, 1.5)
Latex(r'$\lambda_1=%.6f\,\frac{1}{L}, \quad\omega_1^2=%.6f\,\frac{EJ}{mL^4}$'%(l1, l1**4))
```

Out[19]:

$\lambda_1 = 1.302466\,\frac{1}{L},\quad \omega_1^2 = 2.877834\,\frac{EJ}{mL^4}$

Rayleigh Quotient

Using $v = \frac{x}{L}\sin\omega t$ (that is, a rigid rotation about the left hinge) we have

$$T_{\max} = \frac{1}{2}\omega^2\left(\int_0^L m\left(\frac{x}{L}\right)^2 dx + M\,1^2\right) = \frac{1}{2}\omega^2\left(\frac{1}{3}+8\right)mL$$

and

$$V_{\max} = \frac{1}{2}\left(\int_0^L EJ\left(\frac{x}{L}\right)''^2 + k\,1^2\right) = \frac{1}{2}\left(0+24\right)\frac{EJ}{L^3}.$$

Equating the maximum energies and solving for ω^2 gives

$$\omega^2 = \frac{24\,EJ/L^3}{\frac{25}{3}mL} = 3\,\frac{24}{25}\,\frac{EJ}{mL^4} = \dots$$

```
display(Latex(r'$\omega^2_{R00} = %.3f\,\frac{EJ}{mL^4}$'%(3*24/25)))
```

$\omega^2_{R00} = 2.880\,\frac{EJ}{mL^4}$

We can say that the RQ check reinforces our previouos finding...