Dynamics of Structures

September 6, 2018 Written Test

1 2 DoF system — Support Motion

The system is characterized by the nodal degrees of freedom, the structural matrices and the load vector. With $k = {}^{EJ}/_{L^3}$

$$\bar{\mathbf{x}} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \ \bar{\mathbf{M}} = m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \bar{\mathbf{K}} = \frac{3}{14} k \begin{bmatrix} 15 & -20 & 4L \\ -20 & 64 & -24L \\ 4L & -24L & 16L^2 \end{bmatrix}. \ \bar{\mathbf{p}} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} W(t).$$

Of course we must perform a static condensation to have a positive definite mass matrix, using $x \equiv x_d = \{x_1\ x_2\}^T$ and $x_s = x_3$, using the stiffness matrix given in the text (with $\hat{k} = {}^{3k}/_{14}$) it is

$$\mathbf{p} = \mathbf{p}_d - \mathbf{K}_{ds}\mathbf{K}_{ss}^{-1}\mathbf{p}_s = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 4\,\hat{k}L \\ -24\,\hat{k}L \end{Bmatrix} \frac{1}{16\,\hat{k}L^2} W(t) =$$

$$= \begin{Bmatrix} -{}^1/_4 \\ {}^6/_4 \end{Bmatrix} \frac{W(t)}{L} = \begin{Bmatrix} -{}^1/_4 \\ {}^6/_4 \end{Bmatrix} \Delta\, k \sin 2\omega_0 t.$$

The equation of motion is

$$m \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \ddot{\mathbf{x}} + k \begin{bmatrix} 3 & -3 \\ -3 & 6 \end{bmatrix} \mathbf{x} = \begin{Bmatrix} -{}^1/_4 \\ {}^6/_4 \end{Bmatrix} k\Delta \sin 2\omega_0 t.$$

The eigenvalues and the eigenvectors follow from the equation of free vibrations,

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \boldsymbol{\psi} = \mathbf{0}$$

that admits non trivial solutions when

$$\det \begin{pmatrix} 3 - 2\Lambda & -3 \\ -3 & 6 - 3\Lambda \end{pmatrix} = 0, \qquad \Lambda \equiv \lambda^2 = \frac{\omega^2 m}{k} = \frac{\omega^2}{\omega_0^2}.$$

Expanding the determinant we have $6\Lambda^2 - 21\Lambda + 9 = 0$ and it is

$$\Lambda_{1,2} = \frac{21 \mp \sqrt{21^2 - 4\cdot 6\cdot 9}}{2\cdot 6} = \frac{21 \mp 15}{12} = \begin{cases} 1/2 \\ 3 \end{cases}$$

The eigenvectors can be determined using the one of the equations of free vibrations, here the first one

$$(3 - 2\,1/2)\psi_{11} = 3\psi_{21}, \qquad (3 - 2\cdot 3)\psi_{12} = 3\psi_{22}.$$

Using arbitrarily $\psi_{21} = \psi_{12} = 2$ and solving for the remaining unknowns gives

$$\boldsymbol{\Psi} = \begin{bmatrix} +3 & +2 \\ +2 & -2 \end{bmatrix}$$

and this choice leads to a modal mass matrix

$$\mathbf{M}^\star = \boldsymbol{\Psi}^T \mathbf{M}\,\boldsymbol{\Psi} = m \begin{bmatrix} 30 & 0 \\ 0 & 20 \end{bmatrix}.$$

The modal equation of motion follows from the identity $\mathbf{x} = \boldsymbol{\Psi}\,\mathbf{q}$, so that, premultiplying member by member by $\boldsymbol{\Psi}^T$ using $\delta = \Delta \sin 2\omega_0 t$ we have

$$\mathbf{M}^\star \ddot{\mathbf{q}} + \mathbf{K}^\star \mathbf{q} = \frac{1}{4}\,k\delta \boldsymbol{\Psi}^T \begin{Bmatrix} -1 \\ 6 \end{Bmatrix} = \frac{1}{4}\,k\delta \begin{bmatrix} +3 & +2 \\ +2 & -2 \end{bmatrix} \begin{Bmatrix} -1 \\ 6 \end{Bmatrix} = k\delta \begin{Bmatrix} +9/4 \\ -14/4 \end{Bmatrix}.$$

Premultiplying by ${\mathbf{M}^\star}^{-1}$ and writing separately the two equations

$$\begin{aligned} \ddot{q}_1 + 1/2\,\omega_0^2\, q_1 &= +3/40\,\omega_0^2 \Delta \sin 2\omega_0 t \\ \ddot{q}_2 + 3\,\omega_0^2\, q_2 &= -7/40\,\omega_0^2 \Delta \sin 2\omega_0 t \end{aligned}$$

The particular integral for a harmonic excitation,

$$\ddot{q}_i + \lambda_i^2 \omega_0^2\, q_i = \omega_0^2 d_i \sin \lambda_0 \omega_0 t$$

is $\xi_i(t) = c_i \sin \lambda_0 \omega_0 t$; substituting and simplifying

$$c_i\left(\lambda_i^2 - \lambda_0^2\right) = d_i$$

so that, substituting the numerical values and solving, we have

$$c_1 = \frac{+3/40\,\Delta}{1/2 - 4} = -\frac{3}{140}\Delta, \qquad c_2 = \frac{-7/40\,\Delta}{3 - 4} = +\frac{7}{40}\Delta.$$

The general integral is

$$q_i(t) = a_i \sin\lambda_i\omega_0 t + b_i \cos\lambda_i\omega_0 t + c_i \sin\lambda_0\omega_0 t$$

and, starting from rest conditions, it is

$$b_i = 0, \qquad a_i = -c_i\frac{\lambda_0}{\lambda_i}$$

or, with the numerical values,

$$a_1 = -\left(-\frac{3\Delta}{140}\frac{2}{\sqrt{1/2}}\right) = \frac{3\sqrt{2}\Delta}{70} = +0.0606092\,\Delta,$$

$$a_2 = -\left(+\frac{7\Delta}{40}\frac{2}{\sqrt{3}}\right) = -\frac{7\sqrt{3}\Delta}{60} = -0.202073\,\Delta.$$

The displacement x_2 at $t = \pi/\omega_0$ follows from the modal displacements,

$$q_1(\pi/\omega_0) = +0.0606092\,\Delta\sin\pi/\sqrt{2}, \qquad q_2(\pi/\omega_0) = -0.202073\,\Delta\sin\sqrt{3}\pi$$

(taking into account that $\sin 2\omega_0\pi/\omega_0 = \sin 2\pi \equiv 0$) and because it is $x_2 = 2\,(q_1 - q_2)$ we have, substituting the numerical values,

$$x_2(\pi/\omega_0) = -0.205013\,\Delta$$

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