SDOF linear oscillator

Response to Impulsive Loads & Step by Step Methods

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Outline

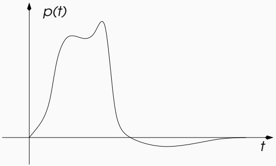
Impulsive Loads

Response to Impulsive Loadings

Nature of Impulsive Loadings

An impulsive load is characterized

- by a single principal impulse, and
- by a relatively short duration.



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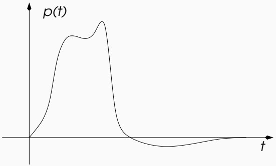

- Impulsive or shock loads are of great importance for the design of certain classes of structural systems, e.g., vehicles or cranes.
- Damping has much less importance in controlling the maximum response to impulsive loadings because the maximum response is reached in a very short time, before the damping forces can dissipate a significant portion of the energy input into the system.
- For this reason, in the following we'll consider only the undamped response to impulsive loads.

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- If $t_0 \ll T_\text{n}$ necessarily R_max happens after the end of the loading, and its value can be determined studying the free vibrations of the dynamic system.
- On the other hand, if the excitation lasts *enough* to have at least a local extreme (maximum or minimum) during the excitation we have to consider the more difficult problem of completely determining the response during the application of the impulsive loading.

Half-sine Wave Impulse

The sine-wave impulse has expression

$$p(t) = \begin{cases} p_0 \sin \dfrac{\pi t}{t_0} = p_0 \sin \omega t & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$

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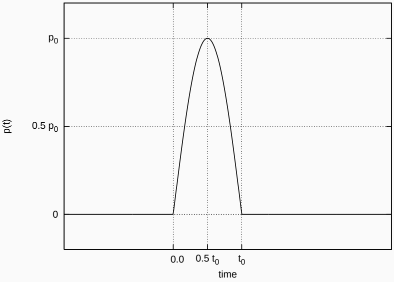

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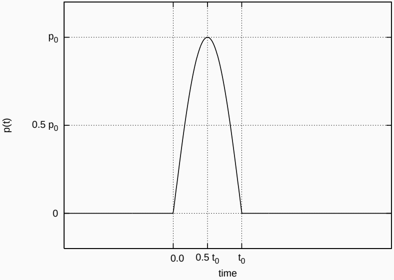

where $\omega = \frac{2\pi}{2t_0}$ is the frequency associated with the load.

Note that $\omega\, t_0 = \pi$.

Response to sine-wave impulse

Consider an undamped *SDOF* initially at rest, with natural period T_n, excited by a half-sine impulse of duration t_0.

The frequency ratio is $\beta = {}^{T_\text{n}}/_{2t_0}$ and the response ratio in the interval $0 < t < t_0$ is

$$R(t) = \frac{1}{1-\beta^2}\,(\sin\omega t - \beta\sin\frac{\omega t}{\beta}),$$

$$\dot{R}(t) = \frac{\omega}{1-\beta^2}\,(\cos\omega t - \cos\frac{\omega t}{\beta}). \qquad [\text{NB: } \frac{\omega}{\beta} = \omega_\text{n}]$$

It is $(1-\beta^2)R(t_0) = -\beta\sin{}^{\pi}/_{\beta}$ and $(1-\beta^2)\dot{R}(t_0) = -\omega\,(1+\cos{}^{\pi}/_{\beta})$, consequently for $t_o \le t$ the response ratio is

$$R(t) = \frac{-\beta}{1-\beta^2}\left((1+\cos\frac{\pi}{\beta})\sin\omega_\text{n}(t-t_0) + \sin\frac{\pi}{\beta}\cos\omega_\text{n}(t-t_0)\right)$$

Maximum response to sine impulse

We have an extreme, and a possible peak value, for $0 \leq t \leq t_0$ if

$$\dot{R}(t) = \frac{\omega}{1-\beta^2}(\cos\omega t - \cos\frac{\omega t}{\beta}) = 0.$$

That implies that $\cos\omega t = \cos{}^{\omega t}/_{\beta} = \cos -{}^{\omega t}/_{\beta}$, whose roots are

$$\omega t = \mp\omega t/\beta + 2n\pi, \; n = 0, \mp1, \mp2, \mp3, \ldots.$$

It is convenient to substitute $\omega t = \pi\alpha$, where $\alpha = t/t_0$:

$$\pi a = \pi\left(\mp\frac{a}{\beta} + 2n\right), \quad n = 0, \mp1, \mp2, \ldots, \quad 0 \leq a \leq 1.$$

Eventually solving for α one has

$$\alpha = \frac{2n\beta}{\beta \pm 1}, \quad n = 0, \mp1, \mp2, \ldots, \quad 0 < \alpha < 1.$$

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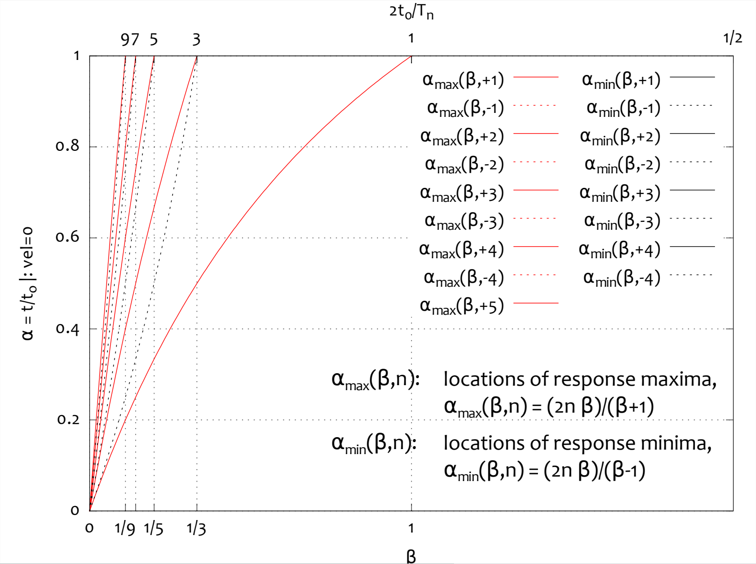
The next slide regards the characteristics of these roots.

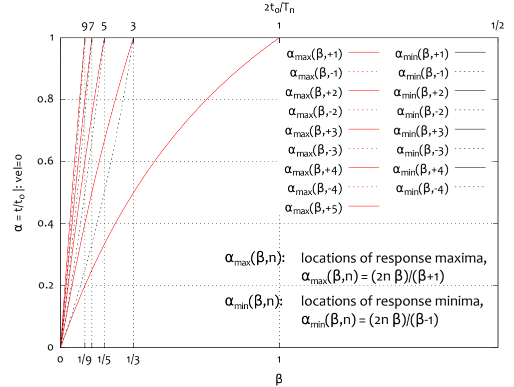
$\alpha(\beta, n)$



2t₀/Tₙ
97 5 3 1 1/2
α = t/t₀ |: vel=0
1
0.8
0.6
0.4
0.2
0
0 1/9 1/5 1/3 1
β
αmax(β,+1) αmin(β,+1)
αmax(β,-1) αmin(β,-1)
αmax(β,+2) αmin(β,+2)
αmax(β,-2) αmin(β,-2)
αmax(β,+3) αmin(β,+3)
αmax(β,-3) αmin(β,-3)
αmax(β,+4) αmin(β,+4)
αmax(β,-4) αmin(β,-4)
αmax(β,+5)
αmax(β,n): locations of response maxima,
αmax(β,n) = (2n β)/(β+1)
αmin(β,n): locations of response minima,
αmin(β,n) = (2n β)/(β-1)

$\alpha(\beta, n)$

In summary, to find the maximum of the response for an assigned $\beta < 1$, one has (a) to compute all $\alpha_k = \frac{2k\beta}{\beta+1}$ until a root is greater than 1, (b) compute all the responses for $t_k = \alpha_k t_0$, (c) choose the maximum of the maxima.



- No roots of type α_{min} for $n > 0$;
- no roots of type α_{max} for $n < 0$;
- no roots for $\beta > 1$, i.e., no roots for $t_0 < \frac{T_\text{n}}{2}$;
- only one root of type α_{max} for $\frac{1}{3} < \beta < 1$, i.e., $\frac{T_\text{n}}{2} < t_0 < \frac{3T_\text{n}}{2}$;
- three roots, two maxima and one minimum, for $\frac{1}{5} < \beta < \frac{1}{3}$;
- five roots, three maxima and two minima, for $\frac{1}{7} < \beta < \frac{1}{5}$;

Maximum response for $\beta > 1$

For $\beta > 1$, the maximum response takes place for $t > t_0$, and its absolute value (see slide *Response to sine-wave impulse*) is

$$R_{\max} = \frac{\beta}{1-\beta^2}\sqrt{(1+\cos\frac{\pi}{\beta})^2 + \sin^2\frac{\pi}{\beta}},$$

using a simple trigonometric identity we can write

$$R_{\max} = \frac{\beta}{1-\beta^2}\sqrt{2+2\cos\frac{\pi}{\beta}}$$

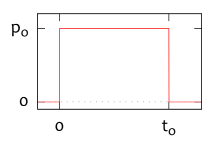
but $1+\cos 2\phi = (\cos^2\phi + \sin^2\phi) + (\cos^2\phi - \sin^2\phi) = 2\cos^2\phi$, so that

$$R_{\max} = \frac{2\beta}{1-\beta^2}\cos\frac{\pi}{2\beta}.$$

Rectangular Impulse

Consider a rectangular impulse of duration t_0,

$$p(t) = p_0 \begin{cases} 1 & \text{for } 0 < t < t_0, \\ 0 & \text{otherwise.} \end{cases}$$



The response ratio and its time derivative are

$$R(t) = 1 - \cos\omega_\text{n} t, \qquad \dot{R}(t) = \omega_\text{n}\sin\omega_\text{n} t,$$

and we recognize that we have maxima $R_\text{max} = 2$ for $\omega_\text{n} t = n\pi$, with the condition $t \le t_0$. Hence we have no maximum during the loading phase for $t_0 < T_\text{n}/2$, and at least one maximum, of value $2\Delta_{st}$, if $t_0 \ge T_\text{n}/2$.

Rectangular Impulse (2)

For shorter impulses, the maximum response ratio is not attained during loading, so we have to compute the amplitude of the free vibrations after the end of loading (remember, as $t_0 \le T_\text{n}/2$ the velocity is positive at $t = t_0$!).

$$R(t) = (1 - \cos\omega_\text{n} t_0)\cos\omega_\text{n}(t - t_0) + (\sin\omega_\text{n} t_0)\sin\omega_\text{n}(t - t_0).$$

The amplitude of the response ratio is then

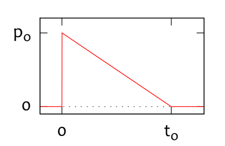
$$A = \sqrt{(1 - \cos\omega_\text{n} t_0)^2 + \sin^2\omega_\text{n} t_0} =$$

$$= \sqrt{2(1 - \cos\omega_\text{n} t_0)} = 2\sin\frac{\omega_\text{n} t_0}{2}.$$

Triangular Impulse

Let's consider the response of a *SDOF* to a triangular impulse,

$$p(t) = p_0\,(1 - t/t_0) \text{ for } 0 < t < t_0$$



As usual, we must start finding the minimum duration that gives place to a maximum of the response in the loading phase, that is

$$R(t) = \frac{1}{\omega_n t_0} \sin \omega_n \frac{t}{t_0} - \cos \omega_n \frac{t}{t_0} + 1 - \frac{t}{t_0}, \quad 0 < t < t_0.$$

Taking the first derivative and setting it to zero, one can see that the first maximum occurs for $t = t_0$ for $t_0 = 0.37101 T_n$, and substituting one can see that $R_{max} = 1$.

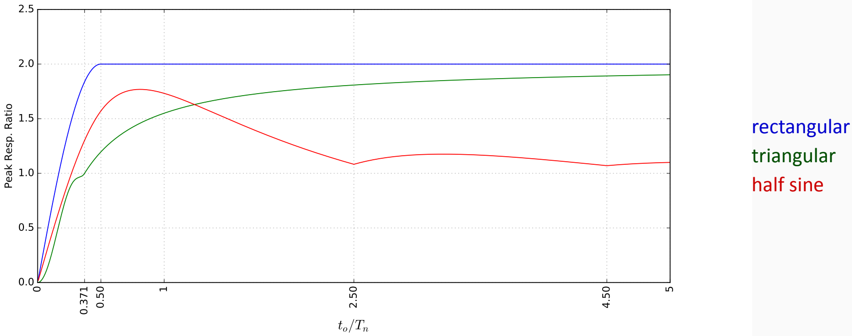
Triangular Impulse (2)

For load durations shorter than $0.37101T_\text{n}$, the maximum occurs after loading and it's necessary to compute the displacement and velocity at the end of the load phase.

For longer loads, the maxima are in the load phase, so that one has to find the all the roots of $\dot{R}(t)$, compute all the extreme values and finally sort out the absolute value maximum.

Shock or response spectra

We have seen that the response ratio is determined by the ratio of the impulse duration to the natural period of the oscillator. One can plot the maximum displacement ratio R_{max} as a function of t_o/T_n for various forms of impulsive loads.



Such plots are commonly known as displacement-response spectra, or simply as response spectra.

Approximate Analysis

For long duration loadings, the maximum response ratio depends on the rate of the increase of the load to its maximum: for a step function we have a maximum response ratio of 2, for a slowly varying load we tend to a quasi-static response, hence a factor $\approxeq 1$

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On the other hand, for short duration loads, the maximum displacement is in the free vibration phase, and its amplitude depends on the work done on the system by the load.
The response ratio depends further on the maximum value of the load impulse, so we can say that the maximum displacement is a more significant measure of response.

Approximate Analysis (2)

An approximate procedure to evaluate the maximum displacement for a short impulse loading is based on the impulse-momentum relationship,

$$m\Delta\dot{x} = \int_0^{t_0} [p(t) - kx(t)]\; dt.$$

When one notes that, for small t_0, the displacement is of the order of t_0^2 while the velocity is in the order of t_0, it is apparent that the kx term may be dropped from the above expression, i.e.,

$$m\Delta\dot{x} \approxeq \int_0^{t_0} p(t)\; dt.$$

Approximate Analysis (3)

Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m}\int_0^{t_0} p(t)\ dt,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \approxeq 0$, one has

$$x(t - t_0) \approxeq \frac{\int_0^{t_0} p(t)\ dt}{m\omega_\mathrm{n}} \sin\omega_\mathrm{n}(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading.

Review

Review of Numerical Methods

Previous Methods

Both the Duhamel integral and the Fourier transform methods lie on on the principle of superposition, i.e., superposition of the responses

- to a succession of infinitesimal impulses, using a convolution (Duhamel) integral, when operating in time domain
- to an infinity of infinitesimal harmonic components, using the frequency response function, when operating in frequency domain.

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The principle of superposition implies *linearity*, but this assumption is often invalid, e.g., a severe earthquake is expected to induce inelastic deformation in a code-designed structure.

State Vector, Linear and Non Linear Systems

The internal state of a linear dynamical system, considering that the mass, the damping and the stiffness do not vary during the excitation, is described in terms of its displacements and its velocity, i.e., the so called *state vector*

$$x = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}.$$

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For a non linear system the state vector must include other information, e.g. the current tangent stiffness, the cumulated plastic deformations, the internal damage, ...

SbS Methods

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Given an initial system state, in step-by-step methods we divide the time in *steps* of known, short duration h_i (usually $h_i = h$, a constant) and from the initial system state at the beginning of each step we compute the final system state at the end of each step.

The final state vector in step i will be the initial state in the subsequent step, $i + 1$.

Step-by-step Methods, 2

Operating independently the analysis for each time step there are no requirements for superposition and non linear behaviour can be considered assuming that the structural properties remain constant during each time step.

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In many cases, the non linear behaviour can be reasonably approximated by a *local* linear model, valid for the duration of the time step.

If the approximation is not good enough, usually a better approximation can be obtained reducing the time step.

Advantages of s-b-s methods

Generality step-by-step methods can deal with every kind of non-linearity, e.g., variation in mass or damping or variation in geometry and not only with mechanical non-linearities.

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Extensibility step-by-step methods can be easily extended to systems with many degrees of freedom, simply using matrices and vectors in place of scalar quantities.

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Errors may be classified as

- phase shifts or change in frequency of the response,
- artificial damping, the numerical procedure removes or adds energy to the dynamic system.

SbS Examples

Examples Of SbS Methods

Response to Impulsive Loading

Review of Numerical Methods

Step-by-step Methods

Examples of SbS Methods

- Piecewise Exact Method
- Central Differences Method
- Methods based on Integration
  - Constant Acceleration Method
  - Linear Acceleration Method

Piecewise exact method

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- We use the exact solution of the equation of motion for a system excited by a linearly varying force, so the source of all errors lies in the piecewise linearisation of the force function and in the approximation due to a local linear model.
- We will see that an appropriate time step can be decided in terms of the number of points required to accurately describe either the force or the response function.

Piecewise exact method

For a generic time step of duration h, consider

- $\{x_0, \dot{x}_0\}$ the initial state vector,
- p_0 and p_1, the values of $p(t)$ at the start and the end of the integration step,
- the linearised force

$$p(\tau) = p_0 + \alpha\tau,\ 0 \le \tau \le h,\ \alpha = (p(h) - p(0))/h,$$

- the forced response

$$x = e^{-\zeta\omega\tau}(A\sin(\omega_\mathrm{D}\tau) + B\cos(\omega_\mathrm{D}\tau)) + (\alpha k\tau + kp_0 - \alpha c)/k^2,$$

where k and c are the stiffness and damping of the SDOF system.

Piecewise exact method

Evaluating the response x and the velocity $\dot{x}$ for $\tau = 0$ and equating to $\{x_0, \dot{x}_0\}$, writing $\Delta_{st} = p(0)/k$ and $\delta(\Delta_{st}) = (p(h) - p(0))/k$, one can find A and B

$$A = \left(\dot{x}_0 + \zeta\omega B - \frac{\delta(\Delta_{st})}{h}\right)\frac{1}{\omega_\mathrm{D}}$$

$$B = x_0 + \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h} - \Delta_{st}$$

substituting and evaluating for $\tau = h$ one finds the state vector at the end of the step.

Piecewise exact method

With

$$S_{\zeta,h} = \sin(\omega_D h)\exp(-\zeta\omega h) \text{ and } C_{\zeta,h} = \cos(\omega_D h)\exp(-\zeta\omega h)$$

and the previous definitions of Δ_{st} and $\delta(\Delta_{st})$, finally we can write

$$x(h) = A\,S_{\zeta,h} + B\,C_{\zeta,h} + (\Delta_{st} + \delta(\Delta_{st})) - \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h}$$

$$\dot{x}(h) = A(\omega_D C_{\zeta,h} - \zeta\omega S_{\zeta,h}) - B(\zeta\omega C_{\zeta,h} + \omega_D S_{\zeta,h}) + \frac{\delta(\Delta_{st})}{h}$$

where

$$B = x_0 + \frac{2\zeta}{\omega}\frac{\delta(\Delta_{st})}{h} - \Delta_{st}, \quad A = \left(\dot{x}_0 + \zeta\omega B - \frac{\delta(\Delta_{st})}{h}\right)\frac{1}{\omega_D}.$$

Example

We have a damped system that is excited by a load in resonance with the system, we know the exact response

$$x(t) = \Delta_{\text{st}}\big((\frac{\sin\omega_D t}{2\sqrt{1-\zeta^2}} + \frac{\cos\omega_D t}{2\zeta})\exp(-\zeta\omega_n t) - \frac{\cos\omega_n t}{2\zeta}\big)$$

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and we want to compute a step-by-step approximation using different step lengths.

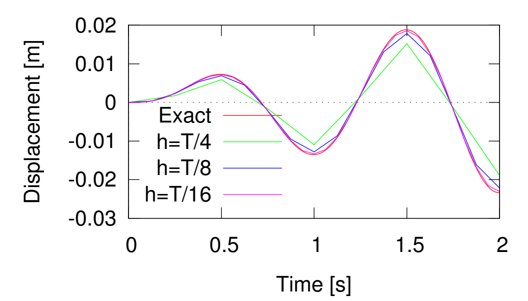
m=1000kg,

k=$4\pi^2$ 1000N/m,

$\omega=2\pi$,

ζ=0.05,

$p(t) =$ $4\pi^2 5$ N $\sin(2\pi\, t)$



It is apparent that you have a very good approximation when the linearised

Central differences

To derive the Central Differences Method, we write the eq. of motion at time $\tau = 0$ and find the initial acceleration,

$$m\ddot{x}_0 + c\dot{x}_0 + kx_0 = p_0 \Rightarrow \ddot{x}_0 = \frac{1}{m}(p_0 - c\dot{x}_0 - kx_0)$$

On the other hand, the initial acceleration can be expressed in terms of finite differences,

$$\ddot{x}_0 = \frac{x_1 - 2x_0 + x_{-1}}{h^2} = \frac{1}{m}(p_0 - c\dot{x}_0 - kx_0)$$

solving for x_1

$$x_1 = 2x_0 - x_{-1} + \frac{h^2}{m}(p_0 - c\dot{x}_0 - kx_0)$$

Central differences

We have an expression for x_1, the displacement at the end of the step,

$$x_1 = 2x_0 - x_{-1} + \frac{h^2}{m}(p_0 - c\dot{x}_0 - kx_0),$$

but we have an additional unknown, x_{-1}... if we write the finite differences approximation to $\dot{x}_0$ we can find an approximation to x_{-1} in terms of the initial velocity $\dot{x}_0$ and the unknown x_1

$$\dot{x}_0 = \frac{x_1 - x_{-1}}{2h} \Rightarrow x_{-1} = x_1 - 2h\dot{x}_0$$

Substituting in the previous equation

$$x_1 = 2x_0 - x_1 + 2h\dot{x}_0 + \frac{h^2}{m}(p_0 - c\dot{x}_0 - kx_0),$$

and solving for x_1

$$x_1 = x_0 + h\dot{x}_0 + \frac{h^2}{2m}(p_0 - c\dot{x}_0 - kx_0)$$

Central differences

$$x_1 = x_0 + h\dot{x}_0 + \frac{h^2}{2m}(p_0 - c\dot{x}_0 - kx_0)$$

To start a new step, we need the value of $\dot{x}_1$, but we may approximate the mean velocity, again, by finite differences

$$\frac{\dot{x}_0 + \dot{x}_1}{2} = \frac{x_1 - x_0}{h} \Rightarrow \dot{x}_1 = \frac{2(x_1 - x_0)}{h} - \dot{x}_0$$

The method is very simple, but it is *conditionally stable*. The stability condition is defined with respect to the natural frequency, or the natural period, of the SDOF oscillator,

$$\omega_\mathrm{n} h \le 2 \Rightarrow h \le \frac{T_n}{\pi} \approx 0.32 T_n$$

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For a SDOF this is not relevant because, as we have seen in our previous example, we need more points for response cycle to correctly represent the response.

Methods based on Integration

We will make use of an *hypothesis* on the variation of the acceleration during the time step and of analytical integration of acceleration and velocity to step forward from the initial to the final condition for each time step.

In general, these methods are based on the two equations

$$\dot{x}_1 = \dot{x}_0 + \int_0^h \ddot{x}(\tau)\, d\tau,$$
$$x_1 = x_0 + \int_0^h \dot{x}(\tau)\, d\tau,$$

which express the final velocity and the final displacement in terms of the initial values x_0 and $\dot{x}_0$ and some definite integrals that depend on the *assumed* variation of the acceleration during the time step.

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- the constant acceleration method,
- the linear acceleration method,
- the family of methods known as *Newmark Beta Methods*, that comprises the previous methods as particular cases.

Constant Acceleration

Here we assume that the acceleration is constant during each time step, equal to the mean value of the initial and final values:

$$\ddot{x}(\tau) = \ddot{x}_0 + \Delta\ddot{x}/2,$$

where $\Delta\ddot{x} = \ddot{x}_1 - \ddot{x}_0$, hence

$$\dot{x}_1 = \dot{x}_0 + \int_0^h (\ddot{x}_0 + \Delta\ddot{x}/2)\, d\tau$$

$$\Rightarrow \Delta\dot{x} = \ddot{x}_0 h + \Delta\ddot{x} h/2$$

$$x_1 = x_0 + \int_0^h (\dot{x}_0 + (\ddot{x}_0 + \Delta\ddot{x}/2)\tau) d\tau$$

$$\Rightarrow \Delta x = \dot{x}_0 h + (\ddot{x}_0) h^2/2 + \Delta\ddot{x} h^2/4$$

Constant acceleration

Taking into account the two equations on the right of the previous slide, and solving for $\Delta\dot{x}$ and $\Delta\ddot{x}$ in terms of Δx, we have

$$\Delta\dot{x} = \frac{2\Delta x - 2h\dot{x}_0}{h}, \quad \Delta\ddot{x} = \frac{4\Delta x - 4h\dot{x}_0 - 2\ddot{x}_0 h^2}{h^2}.$$

We have two equations and three unknowns... Assuming that the system characteristics are constant during a single step, we can write the equation of motion at times $\tau = h$ and $\tau = 0$, subtract member by member and write the *incremental equation of motion*

$$m\Delta\ddot{x} + c\Delta\dot{x} + k\Delta x = \Delta p,$$

that is a third equation that relates our unknowns.

Constant acceleration

Substituting the above expressions for $\Delta\dot{x}$ and $\Delta\ddot{x}$ in the incremental eq. of motion and solving for Δx gives, finally,

$$\Delta x = \frac{\tilde{p}}{\tilde{k}}, \qquad \Delta\dot{x} = \frac{2\Delta x - 2h\dot{x}_0}{h}$$

where

$$\tilde{k} = k + \frac{2c}{h} + \frac{4m}{h^2}$$

$$\tilde{p} = \Delta p + 2c\dot{x}_0 + m(2\ddot{x}_0 + \frac{4}{h}\dot{x}_0)$$

While it is possible to compute the final acceleration in terms of Δx, to achieve a better accuracy it is usually computed solving the equation of equilibrium written at the end of the time step.

Constant Acceleration

Two further remarks

1. The method is *unconditionally stable*
2. The effective stiffness, disregarding damping, is $\tilde{k} \approx k + 4m/h^2$.

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Dividing both members of the above equation by k it is

$$\frac{\tilde{k}}{k} = 1 + \frac{4}{\omega_\text{n}^2 h^2} = 1 + \frac{4}{(2\pi/T_\text{n})^2 h^2} = \frac{T_\text{n}^2}{\pi^2 h^2},$$

The number n_T of time steps in a period T_n is related to the time step duration, $n_\text{T} = T_\text{n}/h$, solving for h and substituting in our last equation, we have

$$\frac{\tilde{k}}{k} \approx 1 + \frac{n_\text{T}^2}{\pi^2}$$

For, e.g., $n_\text{T} = 2\pi$ it is $\tilde{k}/k \approx 1 + 4$, the mass contribution to the effective stiffness is four times the elastic stiffness and the 80% of the total.

Linear Acceleration

We assume that the acceleration is linear, i.e.

$$\ddot{x}(t) = \ddot{x}_0 + \Delta\ddot{x}\frac{\tau}{h}$$

hence

$$\Delta\dot{x} = \ddot{x}_0 h + \Delta\ddot{x} h/2, \quad \Delta x = \dot{x}_0 h + \ddot{x}_0 h^2/2 + \Delta\ddot{x} h^2/6$$

Following a derivation similar to what we have seen in the case of constant acceleration, we can write, again,

$$\Delta x = \left(k + 3\frac{c}{h} + 6\frac{m}{h^2}\right)^{-1}\left[\Delta p + c(\ddot{x}_0\frac{h}{2} + 3\dot{x}_0) + m(3\ddot{x}_0 + 6\frac{\dot{x}_0}{h})\right]$$

$$\Delta\dot{x} = \Delta x\frac{3}{h} - 3\dot{x}_0 - \ddot{x}_0\frac{h}{2}$$

Linear Acceleration

The linear acceleration method is *conditionally stable*, the stability condition being

$$\frac{h}{T} \leq \frac{\sqrt{3}}{\pi} \approx 0.55$$

When dealing with SDOF systems, this condition is never of concern, as we need a shorter step to accurately describe the response of the oscillator, let's say $h \leq 0.12T\ldots$

When stability is not a concern, the accuracy of the linear acceleration method is far superior to the accuracy of the constant acceleration method, so that this is the method of choice for the analysis of SDOF systems.

Newmark Beta Methods

The constant and linear acceleration methods are just two members of the family of Newmark Beta methods, where we write

$$\Delta\dot{x} = (1-\gamma)h\ddot{x}_0 + \gamma h\ddot{x}_1$$
$$\Delta x = h\dot{x}_0 + (\frac{1}{2}-\beta)h^2\ddot{x}_0 + \beta h^2\ddot{x}_1$$

The factor γ weights the influence of the initial and final accelerations on the velocity increment, while β has a similar role with respect to the displacement increment.

Newmark Beta Methods

Using $\gamma \neq 1/2$ leads to numerical damping, so when analysing SDOF systems, one uses $\gamma = 1/2$ (numerical damping may be desirable when dealing with MDOF systems).

Using $\beta = \frac{1}{4}$ leads to the constant acceleration method, while $\beta = \frac{1}{6}$ leads to the linear acceleration method. In the context of MDOF analysis, it's worth knowing what is the minimum β that leads to an unconditionally stable behaviour.

Newmark Beta Methods

The general format for the solution of the incremental equation of motion using the Newmark Beta Method can be written as follows:

$$\Delta x = \frac{\Delta\tilde{p}}{\tilde{k}}$$

$$\Delta v = \frac{\gamma}{\beta}\frac{\Delta x}{h} - \frac{\gamma}{\beta}v_0 + h\left(1 - \frac{\gamma}{2\beta}\right)a_0$$

with

$$\tilde{k} = k + \frac{\gamma}{\beta}\frac{c}{h} + \frac{1}{\beta}\frac{m}{h^2}$$

$$\Delta\tilde{p} = \Delta p + \left(h\left(\frac{\gamma}{2\beta} - 1\right)c + \frac{1}{2\beta}m\right)a_0 + \left(\frac{\gamma}{\beta}c + \frac{1}{\beta}\frac{m}{h}\right)v_0$$

Non Linear Systems

A convenient procedure for integrating the response of a non linear system is based on the incremental formulation of the equation of motion, where for the stiffness and the damping were taken values representative of their variation during the time step: in line of principle, the mean values of stiffness and damping during the time step, or, as this is usually not possible, their initial values, k_0 and c_0.

The Newton-Raphson method can be used to reduce the unbalanced forces at the end of the step.

Modified Newton-Raphson Method

Usually we use the modified Newton-Raphson method, characterised by not updating the system stiffness at each iteration. In pseudo-code, referring for example to the Newmark Beta Method

```
x1,v1,f1 = x0,v0,f0 % initialisation; gb=gamma/beta
Dr = DpTilde
loop:
   Dx = Dr/kTilde
   x2 = x1 + Dx
   v2 = gb*Dx/h - gb*v1 + (1-gb/2)*h*a0
   x_pl = update_u_pl(...)
   f2 = k*(x2-x_pl)
   % important
   Df = (f2-f1) + (kTilde-k_ini)*Dx
   Dr = Dr - Df
   x1, v1, f1 = x2, v2, f2
```

Exercise

A system has a mass $m = 1000$kg, a stiffness $k = 40000$N/m and a viscous damping whose ratio to the critical damping is $\zeta = 0.03$.

The spring is elastoplastic, with a yielding force of 2500N.

The load is an half-sine impulse, with duration 0.3s and maximum value of 6000N.

Use the constant acceleration method to integrate the response, with $h = 0.05$s and, successively, $h = 0.02$s . Note that the stiffness is either 0 or k, write down the expression for the effective stiffness and loading in the incremental formulation, write a spreadsheet or a program to make the computations.