Truncated Sums, Matrix Iteration

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April 3, 2019

Part I

Truncated Sums in Modal Expansions

Eigenvector Expansion
Definitions
Inversion of Eigenvector Expansion

Uncoupled Equations of Motion

Truncated Sum

Eigenvector Expansion

For a N-*DOF* system, it is possible and often advantageous to represent the displacements $\boldsymbol{x}$ in terms of a linear combination of the free vibration modal shapes, the eigenvectors, by the means of a set of modal coordinates,

$$\boldsymbol{x} = \sum_1^N \boldsymbol{\psi}_i q_i = \boldsymbol{\Psi}\boldsymbol{q}.$$

Eigenvector Expansion

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The eigenvectors play a role analogous to the role played by trigonometric functions in Fourier Analysis,

- the eigenvectors possess orthogonality properties,
- the response can be approximated using only a few low frequency terms.

Inverting Eigenvector Expansion

The columns of the eigenmatrix $\boldsymbol{\Psi}$ are the N linearly indipendent eigenvectors $\boldsymbol{\psi}_i$, hence the eigenmatrix is non-singular and it is always correct to write $\boldsymbol{q} = \boldsymbol{\Psi}^{-1}\boldsymbol{x}$.
However, it is not necessary to invert the eigenmatrix...

Inverting Eigenvector Expansion

The modal expansion is

$$\boldsymbol{x} = \sum \boldsymbol{\psi}_i q_i = \boldsymbol{\Psi}\,\boldsymbol{q};$$

multiply each member by $\boldsymbol{\Psi}^T\boldsymbol{M}$, taking into account that $\boldsymbol{M}^\star = \boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{x} = \boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{\Psi}\boldsymbol{q} \qquad \Rightarrow \qquad \boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{x} = \boldsymbol{M}^\star\boldsymbol{q}$$

but $\boldsymbol{M}^\star$ is a diagonal matrix, hence $(\boldsymbol{M}^\star)^{-1} = \{\delta_{ij}/M_i\}$ and we can write

$$\boldsymbol{q} = \boldsymbol{M}^{\star -1}\boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{x}, \qquad \text{or} \qquad q_i = \frac{\boldsymbol{\psi}_i{}^T\boldsymbol{M}\boldsymbol{x}}{M_i}.$$

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Note: this formula works also when we don't know all the eigenvectors and the inversion of a partial, rectangular $\boldsymbol{\Psi}$ is not feasible.

Eigenvector Expansion

Uncoupled Equations of Motion

- Undamped
- Damped System
- Initial Conditions

Truncated Sum

Undamped System

Substituting the modal expansion $\boldsymbol{x} = \boldsymbol{\Psi}\,\boldsymbol{q}$ into the equation of motion, $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{p}(t)$,

$$\boldsymbol{M}\boldsymbol{\Psi}\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{\Psi}\boldsymbol{q} = \boldsymbol{p}(t).$$

Premultiplying each term by $\boldsymbol{\Psi}^T$ and using the orthogonality of the eigenvectors with respect to the structural matrices, for each modal DOF we have an indipendent equation of dynamic equilibrium,

$$M_i\,\ddot{q}_i + \omega_i^2 M_i q_i = p_i^\star(t), \quad i = 1, \ldots, N.$$

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The equations of motion written in terms of nodal coordinates constitute a system of N interdipendent, *coupled* differential equations, written in terms of modal coordinates constitute a set of N indipendent, *uncoupled* differential equations.

Damped System

For a damped system, the equation of motion is

$$\boldsymbol{M}\,\ddot{\boldsymbol{x}} + \boldsymbol{C}\,\dot{\boldsymbol{x}} + \boldsymbol{K}\,\boldsymbol{x} = \boldsymbol{p}(t)$$

and in modal coordinates

$$M_i\,\ddot{q}_i + \boldsymbol{\psi}^T\boldsymbol{C}\,\boldsymbol{\Psi}\,\dot{\boldsymbol{q}} + \omega_i^2 M_i q_i = p_i^\star(t).$$

With $\boldsymbol{\psi}_i^T\boldsymbol{C}\,\boldsymbol{\psi}_j = c_{ij}$ the i-th equation of dynamic equilibrium is

$$M_i\,\ddot{q}_i + \sum_j c_{ij}\,\dot{q}_j + \omega_i^2 M_i q_i = p_i^\star(t), \qquad i = 1, \ldots, N;$$

The equations of motion in modal coordinates are uncoupled only if $c_{ij} = \delta_{ij} C_i$.

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The equations of motion in modal coordinates are uncoupled only if $c_{ij} = \delta_{ij}C_i$.
If we define the damping matrix as

$$\boldsymbol{C} = \sum_b \mathfrak{c}_b \boldsymbol{M}\left(\boldsymbol{M}^{-1}\boldsymbol{K}\right)^b,$$

we know that, as required,

$$c_{ij} = \delta_{ij}C_i \quad \text{with } C_i\ (= 2\zeta_i M_i\omega_i) = \sum_b \mathfrak{c}_b \left(\omega_i^2\right)^b.$$

Damped Systems, a Comment

If the response is computed by modal superposition, it is usually preferred a simpler but equivalent procedure: for each mode of interest the analyst imposes a given damping ratio and the integration of the modal equation of equilibrium is carried out as usual.

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The $\sum \mathfrak{c}_b \ldots$ procedure is useful when, e.g. for non-linear problems, the integration of the eq. of motion is carried out in nodal coordinates, because it is easier to specify damping properties globally as elastic modes properties (that can be measured or deduced from similar outsets) than to assign correct damping properties at the *FE* level and assembling $\boldsymbol{C}$ by the *FEM*.

Initial Conditions

For a damped system, the modal response can be evaluated, for rest initial conditions, using the Duhamel integral,

$$q_i(t) = \frac{1}{M_i\omega_i}\int_0^t p_i(\tau)e^{-\zeta_i\omega_i(t-\tau)}\sin\omega_{Di}(t-\tau)\,\mathrm{d}\tau$$

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For different initial conditions $\boldsymbol{x}_0$, $\dot{\boldsymbol{x}}_0$, we can easily have the initial conditions in modal coordinates:

$$\boldsymbol{q}_0 = \boldsymbol{M}^{\star-1}\boldsymbol{\Psi}^T\boldsymbol{M}\boldsymbol{x}_0$$

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{M}^{\star-1}\boldsymbol{\Psi}^T\boldsymbol{M}\dot{\boldsymbol{x}}_0$$

and the total modal response can be obtained by superposition of Duhamel integral and free vibrations,

$$q_i(t) = e^{-\zeta_i\omega_i t}(q_{i,0}\cos\omega_{Di}t + \frac{\dot{q}_{i,0}+q_{i,0}\zeta_i\omega_i}{\omega_{Di}}\sin\omega_{Di}t) + \cdots$$

Having computed all the N *modal* responses, $q_i(t)$, the response in terms of *nodal* coordinates is the sum of all the N eigenvectors, each multiplied by the corresponding modal response:

$$\begin{aligned} \boldsymbol{x}(t) &= \sum_{i=1}^{N} \boldsymbol{\psi}_i q_i(t) \\ &= \boldsymbol{\psi}_1 q_1(t) + \boldsymbol{\psi}_2 q_2(t) + \cdots + \boldsymbol{\psi}_N q_N(t) \end{aligned}$$

Eigenvector Expansion

Uncoupled Equations of Motion

Truncated Sum
- Definition
- Elastic Forces
- Example

Truncated sum

A *truncated sum* uses only $M < N$ of the lower frequency modes

$$\boldsymbol{x}(t) \approx \sum_{i=1}^{M<N} \boldsymbol{\psi}_i q_i(t),$$

and, under wide assumptions, gives you a good approximation of the structural response.

The importance of truncated sum approximation is twofold:

- less computational effort: less eigenpairs to calculate, less equation of motion to integrate etc
- in FEM models the higher modes are rough approximations to structural ones (mostly due to uncertainties in mass distribution details) and the truncated sum excludes potentially spurious contributions from the response.

Elastic Forces

Until now, we showed interest in displacements only, but we are interested in elastic forces too. We know that elastic forces can be expressed in terms of displacements and the stiffness matrix:

$$\boldsymbol{f}_S(t) = \boldsymbol{K}\,\boldsymbol{x}(t) = \boldsymbol{K}\boldsymbol{\psi}_1 q_1(t) + \boldsymbol{K}\boldsymbol{\psi}_2 q_2(t) + \cdots .$$

From the characteristic equation we know that

$$\boldsymbol{K}\boldsymbol{\psi}_i = \omega_i^2 \boldsymbol{M}\boldsymbol{\psi}_i$$

substituting in the previous equation

$$\boldsymbol{f}_S(t) = \omega_1^2 \boldsymbol{M}\boldsymbol{\psi}_1 q_1(t) + \omega_2^2 \boldsymbol{M}\boldsymbol{\psi}_2 q_2(t) + \cdots .$$

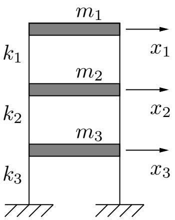
Elastic Forces, 2

The high frequency modes contribution to the elastic forces, e.g.

$$\boldsymbol{f}_S(t) = \omega_1^2 \boldsymbol{M}\boldsymbol{\psi}_1 q_1(t) + \cdots + \omega_{20}^2 \boldsymbol{M}\boldsymbol{\psi}_{20} q_{20}(t) + \cdots,$$

when compared to low frequency mode contributions are more important than their contributions to displacement, because of the multiplicative term ω_i^2.
From this fact follows that, to estimate internal forces within a given accuracy a greater number of modes must be considered in a truncated sum than the number required to estimate displacements within the same accuracy.

Example: problem statement



$$
\begin{aligned}
k_1 &= 120\,\text{MN/m}, & m_1 &= 200\,\text{t},\\
k_2 &= 240\,\text{MN/m}, & m_2 &= 300\,\text{t},\\
k_3 &= 360\,\text{MN/m}, & m_3 &= 400\,\text{t}.
\end{aligned}
$$

1. The above structure is subjected to these initial conditions,

$$
\begin{aligned}
\boldsymbol{x}_0^T &= \{5\,\text{mm} \quad 4\,\text{mm} \quad 3\,\text{mm}\},\\
\dot{\boldsymbol{x}}_0^T &= \{0 \quad 9\,\text{mm/s} \quad 0\}.
\end{aligned}
$$

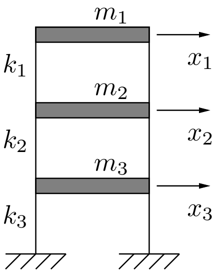
   Write the equation of motion using modal superposition.

2. The above structure is subjected to a half-sine impulse,

$$\boldsymbol{p}^T(t) = \{1 \quad 2 \quad 2\}\, 2.5\,\text{MN} \sin\frac{\pi\, t}{t_1}, \quad \text{with } t_1 = 0.02\,\text{s}.$$

   Write the equation of motion using modal superposition.

Example: structural matrices



$$k_1 = 120\,\text{MN/m},\quad m_1 = 200\,\text{t},$$
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The structural matrices can be written

$$\boldsymbol{K} = k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} = k\overline{\boldsymbol{K}}, \qquad \text{with } k = 120\,\frac{\text{MN}}{\text{m}},$$

$$\boldsymbol{M} = m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = m\overline{\boldsymbol{M}}, \qquad \text{with } m = 100000\,kg.$$

Example: adimensional eigenvalues

We want the solutions of the characteristic equation, so we start writing that the determinant of the equation must be zero:

$$\left\| \overline{\boldsymbol{K}} - \frac{\omega^2}{k/m} \overline{\boldsymbol{M}} \right\| = \left\| \overline{\boldsymbol{K}} - \Omega^2 \overline{\boldsymbol{M}} \right\| = 0,$$

with $\omega^2 = 1200 \left(\frac{\text{rad}}{\text{s}}\right)^2 \Omega^2$.
Expanding the determinant

$$\begin{Vmatrix} 1-2\Omega^2 & -1 & 0 \\ -1 & 3-3\Omega^2 & -2 \\ 0 & -2 & 5-4\Omega^2 \end{Vmatrix} = 0$$

we have the following algebraic equation of 3rd order in Ω^2

$$24 \left(\Omega^6 - \frac{11}{4}\Omega^4 + \frac{15}{8}\Omega^2 - \frac{1}{4} \right) = 0.$$

Example: table of eigenvalues etc

Here are the adimensional roots Ω_i^2, $i = 1, 2, 3$, the dimensional eigenvalues $\omega_i^2 = 1200\frac{\text{rad}^2}{\text{s}^2}\Omega_i^2$ and all the derived dimensional quantities:

$$
\begin{aligned}
\Omega_1^2 &= 0.17573 & \Omega_2^2 &= 0.8033 & \Omega_3^2 &= 1.7710 \\
\omega_1^2 &= 210.88 & \omega_2^2 &= 963.96 & \omega_3^2 &= 2125.2 \\
\omega_1 &= 14.522 & \omega_2 &= 31.048 & \omega_3 &= 46.099 \\
f_1 &= 2.3112 & f_2 &= 4.9414 & f_3 &= 7.3370 \\
T_1 &= 0.43268 & T_3 &= 0.20237 & T_3 &= 0.1363
\end{aligned}
$$

Example: eigenvectors and modal matrices

With $\psi_{1j} = 1$, using the 2nd and 3rd equations,

$$\begin{bmatrix} 3 - 3\Omega_j^2 & -2 \\ -2 & 5 - 4\Omega_j^2 \end{bmatrix} \begin{Bmatrix} \psi_{2j} \\ \psi_{3j} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

The above equations must be solved for $j = 1, 2, 3$. The solutions are finally collected in the eigenmatrix

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & 1 & 1 \\ +0.648535272183 & -0.606599092464 & -2.54193617967 \\ +0.301849953585 & -0.678977475113 & +2.43962752148 \end{bmatrix}.$$

The Modal Matrices are

$$\boldsymbol{M}^\star = \begin{bmatrix} 362.6 & 0 & 0 \\ 0 & 494.7 & 0 \\ 0 & 0 & 4519.1 \end{bmatrix} \times 10^3\,\text{kg},$$

$$\boldsymbol{K}^\star = \begin{bmatrix} 76.50 & 0 & 0 \\ 0 & 477.0 & 0 \\ 0 & 0 & 9603.9 \end{bmatrix} \times 10^6\,\frac{\text{N}}{\text{m}}$$

Example: initial conditions in modal coordinates

$$\boldsymbol{q}_0 = (\boldsymbol{M}^\star)^{-1}\boldsymbol{\Psi}^T\boldsymbol{M}\begin{Bmatrix}5\\4\\3\end{Bmatrix}\,\mathsf{mm} = \begin{Bmatrix}+5.9027\\-1.0968\\+0.1941\end{Bmatrix}\,\mathsf{mm},$$

$$\dot{\boldsymbol{q}}_0 = (\boldsymbol{M}^\star)^{-1}\boldsymbol{\Psi}^T\boldsymbol{M}\begin{Bmatrix}0\\9\\0\end{Bmatrix}\,\frac{\mathsf{mm}}{\mathsf{s}} = \begin{Bmatrix}+4.8288\\-3.3101\\-1.5187\end{Bmatrix}\,\frac{\mathsf{mm}}{\mathsf{s}}$$

Example: structural response

These are the displacements, in mm

$$x_1 = +5.91\cos(14.5t + .06) + 1.10\cos(31.0t - 3.04) + 0.20\cos(46.1t - 0.17)$$
$$x_2 = +3.83\cos(14.5t + .06) - 0.67\cos(31.0t - 3.04) - 0.50\cos(46.1t - 0.17)$$
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and these the elastic/inertial forces, in kN

$$x_1 = +249.\cos(14.5t + .06) + 212.\cos(31.0t - 3.04) + 084.\cos(46.1t - 0.17)$$
$$x_2 = +243.\cos(14.5t + .06) - 193.\cos(31.0t - 3.04) - 319.\cos(46.1t - 0.17)$$
$$x_3 = +151.\cos(14.5t + .06) - 288.\cos(31.0t - 3.04) + 408.\cos(46.1t - 0.17)$$

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As expected, the contributions of the higher modes are more important for the forces, less important for the displacements.

Part II

Matrix Iteration Procedures

Introduction

Fundamental Mode Analysis

Second Mode Analysis

Higher Modes

Inverse Iteration

Matrix Iteration with Shifts

Rayleigh Methods

Introduction

Dynamic analysis of MDOF systems based on modal superposition is both simple and efficient

- simple: the modal response can be easily computed, analitically or numerically, with the techniques we have seen for SDOF systems,
- efficient: in most cases, only the modal responses of a few lower modes are required to accurately describe the structural response.

Introduction

The structural matrices being easily assembled using the *FEM*, the modal superposition procedure is ready to be applied to structures with thousands, millions of *DOF*'s!

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But wait, we can know how to compute the eigenpairs only when the analyzed structure has very few degrees of freedom...

We will discuss how it is possible to compute the eigenpairs of arbitrarily large dynamic systems using the so called *Matrix Iteration* procedure (and a number of variations derived from this fundamental idea).

Introduction

Fundamental Mode Analysis
- A Possible Procedure
- Matrix Iteration Procedure
- Convergence of Matrix Iteration Procedure

Second Mode Analysis

Higher Modes

Inverse Iteration

Matrix Iteration with Shifts

Rayleigh Methods

Equilibrium

First, we will see an iterative procedure whose outputs are the first, or fundamental, mode shape vector and the corresponding eigenvalue.
When an undamped system freely vibrates with a harmonic time dependency of frequency ω_i, the equation of motion, simplifying the time dependency, is

$$\boldsymbol{K}\,\boldsymbol{\psi}_i = \omega_i^2 \boldsymbol{M}\,\boldsymbol{\psi}_i.$$

In equilibrium terms, the elastic forces are equal to the inertial forces when the systems oscillates with frequency ω_i^2 and mode shape $\boldsymbol{\psi}_i$

Proposal of an iterative procedure

Our iterative procedure will be based on finding a new displacement vector $\boldsymbol{x}_{n+1}$ such that the elastic forces $\boldsymbol{f}_S = \boldsymbol{K}\,\boldsymbol{x}_{i+1}$ are in equilibrium with the inertial forces due to the *old* displacement vector $\boldsymbol{x}_n$, $\boldsymbol{f}_I = \omega_i^2 \boldsymbol{M}\,\boldsymbol{x}_n$, that is

$$\boldsymbol{K}\,\boldsymbol{x}_{n+1} = \omega_i^2 \boldsymbol{M}\,\boldsymbol{x}_n.$$

Premultiplying by the inverse of $\boldsymbol{K}$ and introducing the *Dynamic Matrix*, $\boldsymbol{D} = \boldsymbol{K}^{-1}\boldsymbol{M}$

$$\boldsymbol{x}_{n+1} = \omega_i^2 \boldsymbol{K}^{-1}\boldsymbol{M}\,\boldsymbol{x}_n = \omega_i^2 \boldsymbol{D}\,\boldsymbol{x}_n.$$

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Premultiplying by the inverse of $\boldsymbol{K}$ and introducing the *Dynamic Matrix*, $\boldsymbol{D} = \boldsymbol{K}^{-1}\boldsymbol{M}$

$$\boldsymbol{x}_{n+1} = \omega_i^2 \boldsymbol{K}^{-1}\boldsymbol{M}\,\boldsymbol{x}_n = \omega_i^2 \boldsymbol{D}\,\boldsymbol{x}_n.$$

In the generative equation above we miss a fundamental part, the square of the free vibration frequency ω_i^2.

The Matrix Iteration Procedure, 1

This problem is solved considering the $\boldsymbol{x}_n$ as a sequence of *normalized* vectors and introducing the idea of an *unnormalized* new displacement vector, $\hat{\boldsymbol{x}}_{n+1}$,

$$\hat{\boldsymbol{x}}_{n+1} = \boldsymbol{D}\,\boldsymbol{x}_n,$$

note that we removed the explicit dependency on ω_i^2.

The Matrix Iteration Procedure, 2

The normalized vector is obtained applying to $\hat{\boldsymbol{x}}_{n+1}$ a normalizing factor, $\mathfrak{F}_{n+1}$,

$$\boldsymbol{x}_{n+1} = \frac{\hat{\boldsymbol{x}}_{n+1}}{\mathfrak{F}_{n+1}},$$

but $$\boldsymbol{x}_{n+1} = \omega_i^2 \boldsymbol{D}\, \boldsymbol{x}_n = \omega_i^2\, \hat{\boldsymbol{x}}_{n+1}, \quad \Rightarrow \quad \frac{1}{\mathfrak{F}} = \omega_i^2$$

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If we agree that, near convergence, $\boldsymbol{x}_{n+1} \approx \boldsymbol{x}_n$, substituting in the previous equation we have

$$\boldsymbol{x}_{n+1} \approx \boldsymbol{x}_n = \omega_i^2\, \hat{\boldsymbol{x}}_{n+1} \quad \Rightarrow \quad \omega_i^2 \approx \frac{\boldsymbol{x}_n}{\hat{\boldsymbol{x}}_{n+1}}.$$

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Of course the division of two vectors is not an option, so we want to twist it into something useful.

Normalization

First, consider $\boldsymbol{x}_n = \boldsymbol{\psi}_i$: in this case, for $j = 1, \ldots, N$ it is

$$x_{n,j}/\hat{x}_{n+1,j} = \omega_i^2.$$

When $\boldsymbol{x}_n \neq \boldsymbol{\psi}_i$ it is possible to demonstrate that we can bound the eigenvalue

$$\min_{j=1,\ldots,N} \left\{ \frac{x_{n,j}}{\hat{x}_{n+1,j}} \right\} \leq \omega_i^2 \leq \max_{j=1,\ldots,N} \left\{ \frac{x_{n,j}}{\hat{x}_{n+1,j}} \right\}.$$

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$$\min_{j=1,\ldots,N}\left\{\frac{x_{n,j}}{\hat{x}_{n+1,j}}\right\} \leq \omega_i^2 \leq \max_{j=1,\ldots,N}\left\{\frac{x_{n,j}}{\hat{x}_{n+1,j}}\right\}.$$

A more rational approach would make reference to a proper vector norm, so using our preferred vector norm we can write

$$\omega_i^2 \approx \frac{\hat{\boldsymbol{x}}_{n+1}^T \boldsymbol{M}\,\boldsymbol{x}_n}{\hat{\boldsymbol{x}}_{n+1}^T \boldsymbol{M}\,\hat{\boldsymbol{x}}_{n+1}},$$

Normalization

First, consider $\boldsymbol{x}_n = \boldsymbol{\psi}_i$: in this case, for $j = 1, \ldots, N$ it is

$$x_{n,j}/\hat{x}_{n+1,j} = \omega_i^2.$$

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$$\omega_i^2 \approx \frac{\hat{\boldsymbol{x}}_{n+1}^T \boldsymbol{M}\, \boldsymbol{x}_n}{\hat{\boldsymbol{x}}_{n+1}^T \boldsymbol{M}\, \hat{\boldsymbol{x}}_{n+1}},$$

(if memory helps, this is equivalent to the R_{11} approximation, that we introduced studying Rayleigh quotient refinements).

Proof of Convergence, 1

Until now we postulated that the sequence $\boldsymbol{x}_n$ converges to some, unspecified eigenvector $\boldsymbol{\psi}_i$, now we will demonstrate that the sequence converge to the first, or fundamental mode shape,

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{\psi}_1.$$

1. Expand $\boldsymbol{x}_0$ in terms of eigenvectors an modal coordinates:

$$\boldsymbol{x}_0 = \boldsymbol{\psi}_1 q_{1,0} + \boldsymbol{\psi}_2 q_{2,0} + \boldsymbol{\psi}_3 q_{3,0} + \cdots .$$

2. The inertial forces, assuming that the system is vibrating according to the fundamental frequency, are

$$\begin{aligned}\boldsymbol{f}_{I,n=0} &= \omega_1^2 \boldsymbol{M} \left(\boldsymbol{\psi}_1 q_{1,0} + \boldsymbol{\psi}_2 q_{2,0} + \boldsymbol{\psi}_3 q_{3,0} + \cdots\right) \\ &= \boldsymbol{M} \left(\omega_1^2 \boldsymbol{\psi}_1 q_{1,0} \frac{\omega_1^2}{\omega_1^2} + \omega_2^2 \boldsymbol{\psi}_2 q_{2,0} \frac{\omega_1^2}{\omega_2^2} + \cdots\right).\end{aligned}$$

Proof of Convergence, 2

3. The deflections due to these forces (no hat!, we have multiplied by ω_1^2) are

$$\boldsymbol{x}_{n=1} = \boldsymbol{K}^{-1}\boldsymbol{M}\left(\omega_1^2\boldsymbol{\psi}_1 q_{1,0}\frac{\omega_1^2}{\omega_1^2} + \omega_2^2\boldsymbol{\psi}_2 q_{2,0}\frac{\omega_1^2}{\omega_2^2} + \cdots\right),$$

(note that every term has been multiplied and divided by the corresponding eigenvalue ω_i^2).

Proof of Convergence, 3

4. With $\omega_j^2 \boldsymbol{M}\,\boldsymbol{\psi}_j = \boldsymbol{K}\boldsymbol{\psi}_j$, substituting and simplifying $\boldsymbol{K}^{-1}\boldsymbol{K} = \boldsymbol{I}$,

$$\begin{aligned}\boldsymbol{x}_{n=1} = \boldsymbol{K}^{-1}\Bigg(&\boldsymbol{K}\boldsymbol{\psi}_1 q_{1,0}\left(\frac{\omega_1^2}{\omega_1^2}\right)^1 + \\ &\boldsymbol{K}\boldsymbol{\psi}_2 q_{2,0}\left(\frac{\omega_1^2}{\omega_2^2}\right)^1 + \\ &\boldsymbol{K}\boldsymbol{\psi}_3 q_{3,0}\left(\frac{\omega_1^2}{\omega_3^2}\right)^1 + \cdots\Bigg) \\ = {}&\boldsymbol{\psi}_1 q_{1,0}\frac{\omega_1^2}{\omega_1^2} + \boldsymbol{\psi}_2 q_{2,0}\frac{\omega_1^2}{\omega_2^2} + \boldsymbol{\psi}_3 q_{3,0}\frac{\omega_1^2}{\omega_3^2} + \cdots,\end{aligned}$$

Proof of Convergence, 4

5. applying again this procedure

$$\boldsymbol{x}_{n=2} = \left(\boldsymbol{\psi}_1 q_{1,0}\left(\frac{\omega_1^2}{\omega_1^2}\right)^2 + \boldsymbol{\psi}_2 q_{2,0}\left(\frac{\omega_1^2}{\omega_2^2}\right)^2 + \boldsymbol{\psi}_3 q_{3,0}\left(\frac{\omega_1^2}{\omega_3^2}\right)^2 + \cdots\right),$$

6. applying the procedure n times

$$\boldsymbol{x}_{n} = \left(\boldsymbol{\psi}_1 q_{1,0}\left(\frac{\omega_1^2}{\omega_1^2}\right)^n + \boldsymbol{\psi}_2 q_{2,0}\left(\frac{\omega_1^2}{\omega_2^2}\right)^n + \boldsymbol{\psi}_3 q_{3,0}\left(\frac{\omega_1^2}{\omega_3^2}\right)^n + \cdots\right).$$

Proof of Convergence, 5

Going to the limit,

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{\psi}_1 q_{1,0}$$

because

$$\lim_{n\to\infty} \left(\frac{\omega_1^2}{\omega_j^2}\right)^n = \delta_{1j}$$

Consequently,

$$\lim_{n\to\infty} \frac{|\boldsymbol{x}_n|}{|\hat{\boldsymbol{x}}_n|} = \omega_1^2$$

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- Sweeping Matrix

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Purified Vectors

If we know $\boldsymbol{\psi}_1$ and ω_1^2 from the matrix iteration procedure it is possible to compute the second eigenpair, following a slightly different procedure.

Purified Vectors

If we know ψ_1 and ω_1^2 from the matrix iteration procedure it is possible to compute the second eigenpair, following a slightly different procedure. Express the initial iterate in terms of the (unknown) eigenvectors,

$$\boldsymbol{x}_{n=0} = \boldsymbol{\Psi}\,\boldsymbol{q}_{n=0}$$

and premultiply by the (known) $\boldsymbol{\psi}_1^T \boldsymbol{M}$:

$$\boldsymbol{\psi}_1^T \boldsymbol{M}\,\boldsymbol{x}_{n=0} = M_1 q_{1,n=0}$$

solving for $q_{1,n=0}$

$$q_{1,n=0} = \frac{\boldsymbol{\psi}_1^T \boldsymbol{M}\,\boldsymbol{x}_{n=0}}{M_1}.$$

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solving for $q_{1,n=0}$

$$q_{1,n=0} = \frac{\boldsymbol{\psi}_1^T \boldsymbol{M}\, \boldsymbol{x}_{n=0}}{M_1}.$$

Knowing the amplitude of the 1st modal contribution to $\boldsymbol{x}_{n=0}$ we can write a *purified* vector,

$$\boldsymbol{y}_{n=0} = \boldsymbol{x}_{n=0} - \boldsymbol{\psi}_1 q_{1,n=0}.$$

Convergence (?)

It is easy to demonstrate that using $\boldsymbol{y}_{n=0}$ as our starting vector

$$\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{\psi}_2 q_{2,n=0}, \qquad \lim_{n\to\infty} \frac{|\boldsymbol{y}_n|}{|\hat{\boldsymbol{y}}_n|} = \omega_2^2.$$

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Due to numerical errors in the determination of fundamental mode and in the procedure itself, using a plain matrix iteration the procedure however converges to the 1st eigenvector, so to preserve convergence to the 2nd mode it is necessary that the iterated vector $\boldsymbol{y}_n$ is purified at each step n.

Purification Procedure

The purification procedure is simple, at each step the amplitude of the 1st mode is first computed, then removed from the iterated vector $\boldsymbol{y}_n$

$$q_{1,n} = \boldsymbol{\psi}_1^T \boldsymbol{M} \boldsymbol{y}_n / M_1,$$

$$\hat{\boldsymbol{y}}_{n+1} = \boldsymbol{D} \left(\boldsymbol{y}_n - \boldsymbol{\psi}_1 q_{1,n} \right) = \boldsymbol{D} \left(\boldsymbol{I} - \frac{1}{M_1} \boldsymbol{\psi}_1 \boldsymbol{\psi}_1^T \boldsymbol{M} \right) \boldsymbol{y}_n$$

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Introducing the *sweeping matrix* $\boldsymbol{S}_1 = \boldsymbol{I} - \frac{1}{M_1}\boldsymbol{\psi}_1\boldsymbol{\psi}_1^T\boldsymbol{M}$ and the modified dynamic matrix $\boldsymbol{D}_2 = \boldsymbol{D}\boldsymbol{S}_1$, we can write

$$\hat{\boldsymbol{y}}_{n+1} = \boldsymbol{D}\boldsymbol{S}_1\boldsymbol{y}_n = \boldsymbol{D}_2\boldsymbol{y}_n.$$

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$$\hat{\boldsymbol{y}}_{n+1} = \boldsymbol{D}\boldsymbol{S}_1 \boldsymbol{y}_n = \boldsymbol{D}_2 \boldsymbol{y}_n.$$

This is known as *matrix iteration with sweeps*.

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Third Mode

Using again the idea of purifying the iterated vector, starting with the knowledge of the first and the second eigenpair,

$$\hat{\boldsymbol{y}}_{n+1} = \boldsymbol{D}\left(\boldsymbol{y}_n - \boldsymbol{\psi}_1 q_{1,n} - \boldsymbol{\psi}_2 q_{2,n}\right)$$

with $q_{n,1}$ as before and

$$q_{2,n} = \boldsymbol{\psi}_2^T \boldsymbol{M} \boldsymbol{y}_n / M_2,$$

substituting in the expression for the purified vector

$$\hat{\boldsymbol{y}}_{n+1} = \boldsymbol{D}\big(\underbrace{\boldsymbol{I} - \frac{1}{M_1}\boldsymbol{\psi}_1\boldsymbol{\psi}_1^T\boldsymbol{M}}_{\boldsymbol{S}_1} - \frac{1}{M_2}\boldsymbol{\psi}_2\boldsymbol{\psi}_2^T\boldsymbol{M}\big)\boldsymbol{y}_n$$

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The conclusion is that the sweeping matrix and the modified dynamic matrix to be used to compute the 3rd eigenvector are

$$\boldsymbol{S}_2 = \boldsymbol{S}_1 - \frac{1}{M_2}\boldsymbol{\psi}_2\boldsymbol{\psi}_2^T\boldsymbol{M}, \qquad \boldsymbol{D}_3 = \boldsymbol{D}\,\boldsymbol{S}_2.$$

Generalization to Higher Modes

The results obtained for the third mode are easily generalised.
It is easy to verify that the following procedure can be used to compute all the modes.

Define $\boldsymbol{S}_0 = \boldsymbol{I}$, take $i = 1$,

1. compute the modified dynamic matrix to be used for mode i,
$$\boldsymbol{D}_i = \boldsymbol{D}\,\boldsymbol{S}_{i-i}$$
2. compute $\boldsymbol{\psi}_i$ using the modified dynamic matrix;
3. compute the modal mass $M_i = \boldsymbol{\psi}^T \boldsymbol{M}\,\boldsymbol{\psi}$;
4. compute the sweeping matrix $\boldsymbol{S}_i$ that *sweeps* the contributions of the first i modes from trial vectors,
$$\boldsymbol{S}_i = \boldsymbol{S}_{i-1} - \frac{1}{M_i}\boldsymbol{\psi}_i\boldsymbol{\psi}_i^T\boldsymbol{M};$$
5. increment i, `GOTO 1`.

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5. increment i, `GOTO 1`.

Well, we finally have a method that can be used to compute all the eigenpairs of our dynamic problems, full circle!

Discussion

The method of matrix iteration with sweeping is not used in production because

1. $\boldsymbol{D}$ is a full matrix, even if $\boldsymbol{M}$ and $\boldsymbol{K}$ are banded matrices, and the matrix product that is the essential step in every iteration is computationally onerous,
2. the procedure is however affected by numerical errors,

so, after having demonstrated that it is possible to compute all the eigenvectors of a large problem using an iterative procedure it is time to look for different, more efficient iterative procedures.

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Introduction to Inverse Iteration

Inverse iteration is based on the fact that the symmetric stiffness matrix has a banded structure, that is a relatively large triangular portion of the matrix is composed by zeroes.

The banded structure is due to the FEM model: in every equation of equilibrium the only non zero elastic force coefficients are due to the degrees of freedom of the few FE's that contain the degree of freedom for which the equilibrium is written.

Definition of LU decomposition

Every symmetric, banded matrix can be subjected to a so called LU decomposition, that is, for $\boldsymbol{K}$ we write

$$\boldsymbol{K} = \boldsymbol{L}\boldsymbol{U}$$

where $\boldsymbol{L}$ and $\boldsymbol{U}$ are, respectively, a lower- and an upper-banded matrix. If we denote with b the bandwidth of $\boldsymbol{K}$, we have

$$\boldsymbol{L} = \left[l_{ij}\right] \quad \text{with } l_{ij} \equiv 0 \text{ for } \begin{cases} i < j \\ j < i - b \end{cases}$$

and

$$\boldsymbol{U} = \left[u_{ij}\right] \quad \text{with } u_{ij} \equiv 0 \text{ for } \begin{cases} i > j \\ j > i + b \end{cases}$$

Twice the equations?

In this case, with $\boldsymbol{w}_n = \boldsymbol{M}\,\boldsymbol{x}_n$, the recursion can be written

$$\boldsymbol{L}\,\boldsymbol{U}\,\boldsymbol{x}_{n+1} = \boldsymbol{w}_n$$

or as a system of equations,

$$\boldsymbol{U}\,\boldsymbol{x}_{n+1} = \boldsymbol{z}_{n+1}$$
$$\boldsymbol{L}\,\boldsymbol{z}_{n+1} = \boldsymbol{w}_n$$

Apparently, we have doubled the number of unknowns, but the z_j's can be easily computed by the procedure of *back substitution*.

Back Substitution

Temporarily dropping the n and $n+1$ subscripts, we can write

$$\begin{aligned}
z_1 &= (w_1)/l_{11} \\
z_2 &= (w_2 - l_{21}z_1)/l_{22} \\
z_3 &= (w_3 - l_{31}z_1 - l_{32}z_2)/l_{33} \\
&\dots \\
z_i &= (w_i - \sum_{j=i-b}^{i-1} l_{ij}z_j)/l_{ii} \\
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\end{aligned}$$

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The $\boldsymbol{x}$ are then given by $\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{z}$.

Back Substitution

We have computed $\boldsymbol{z}$ by back substitution, we must solve $\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{z}$ but U is upper triangular, so we have

$$
\begin{aligned}
x_N &= (z_N)/u_{NN} \\
x_{N-1} &= (z_{N-1} - u_{N-1,N} z_N)/u_{N-1,N-1} \\
x_{N-2} &= (z_{N-2} - u_{N-2,N} z_N - u_{N-2,N-1} z_{N-1})/u_{N-2,N-2} \\
&\dots \\
x_{N-j} &= (z_{N-j} - \sum_{k=0}^{j-1} u_{N-j,N-k} z_{N-k})/u_{N-j,N-j},
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&\ldots \\
x_{N-j} &= (z_{N-j} - \sum_{k=0}^{j-1} u_{N-j,N-k} z_{N-k})/u_{N-j,N-j},
\end{aligned}
$$

For moderately large systems, the reduction in operations count given by back substitution with respect to matrix multiplication is so large that the additional cost of the LU decomposition is negligible.

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Introduction to Shifts

Inverse iteration can be applied to each step of matrix iteration with sweeps, or to each step of a different procedure intended to compute all the eigenpairs, the *matrix iteration with shifts*.

Matrix Iteration with Shifts, 1

If we write

$$\omega_i^2 = \mu + \lambda_i,$$

where μ is a *shift* and λ_i is a *shifted eigenvalue*, the eigenvalue problem can be formulated as

$$\boldsymbol{K}\,\boldsymbol{\psi}_i = (\mu + \lambda_i)\boldsymbol{M}\,\boldsymbol{\psi}_i$$

or

$$(\boldsymbol{K} - \mu\boldsymbol{M})\boldsymbol{\psi}_i = \lambda_i\boldsymbol{M}\,\boldsymbol{\psi}_i.$$

If we introduce a modified stiffness matrix

$$\overline{\boldsymbol{K}} = \boldsymbol{K} - \mu\boldsymbol{M},$$

we recognize that we have a *new* problem, that has *exactly* the same eigenvectors and *shifted* eigenvalues,

$$\overline{\boldsymbol{K}}\,\boldsymbol{\phi}_i = \lambda_i\boldsymbol{M}\boldsymbol{\phi}_i,$$

where

$$\boldsymbol{\phi}_i = \boldsymbol{\psi}_i, \qquad \lambda_i = \omega_i^2 - \mu.$$

Matrix Iteration with Shifts, 2

The shifted eigenproblem can be solved, e.g., by matrix iteration and the procedure will converge to the *smallest absolute value* shifted eigenvalue and to the associated eigenvector. After convergence is reached,

$$\boldsymbol{\psi}_i = \boldsymbol{\phi}_i, \qquad \omega_i^2 = \lambda_i + \mu.$$

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$$\boldsymbol{\psi}_i = \boldsymbol{\phi}_i, \qquad \omega_i^2 = \lambda_i + \mu.$$

The convergence of the method can be greatly enhanced if the shift μ is updated every few steps during the iterative procedure using the current best estimate of λ_i,

$$\lambda_{i,n+1} = \frac{\hat{\boldsymbol{x}}_{n+1} \boldsymbol{M}\, \boldsymbol{x}_n}{\hat{\boldsymbol{x}}_{n+1} \boldsymbol{M}\, \hat{\boldsymbol{x}}_{n+1}},$$

to improve the modified stiffness matrix to be used in the following iterations,

$$\overline{\boldsymbol{K}} = \overline{\boldsymbol{K}} - \lambda_{i,n+1} \boldsymbol{M}$$

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to improve the modified stiffness matrix to be used in the following iterations,

$$\overline{\boldsymbol{K}} = \overline{\boldsymbol{K}} - \lambda_{i,n+1} \boldsymbol{M}$$

Much thought was spent on the problem of choosing the initial shifts, so that all the eigenvectors can be computed in sequence without missing any of them.

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Rayleigh Quotient for Discrete Systems

The matrix iteration procedures are usually used in conjunction with methods derived from the Rayleigh Quotient method.

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The procedure to be used for discrete systems can be summarized as

$$\boldsymbol{x}(t) = \boldsymbol{\phi} Z_0 \sin\omega t, \qquad \dot{\boldsymbol{x}}(t) = \omega\boldsymbol{\phi} Z_0 \cos\omega t,$$

$$2T_{\textsf{max}} = \omega^2 \boldsymbol{\phi}^T \boldsymbol{M}\,\boldsymbol{\phi}, \qquad 2V_{\textsf{max}} = \boldsymbol{\phi}^T \boldsymbol{K}\,\boldsymbol{\phi},$$

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equating the maxima, we have

$$\omega^2 = \frac{\boldsymbol{\phi}^T \boldsymbol{K}\, \boldsymbol{\phi}}{\boldsymbol{\phi}^T \boldsymbol{M}\, \boldsymbol{\phi}} = \frac{k^\star}{m^\star},$$

where $\boldsymbol{\phi}$ is an assumed shape vector, not an eigenvector.

Ritz Coordinates

For a N *DOF* system, an *approximation* to a displacement vector $\boldsymbol{x}$ can be written in terms of a set of $M < N$ assumed shape, linearly independent vectors,

$$\boldsymbol{\phi}_i, \quad i = 1, \ldots, M < N$$

and a set of *Ritz coordinates* z_i, $i - 1, \ldots, M < N$:

$$\boldsymbol{x} = \sum_i \boldsymbol{\phi}_i z_i = \boldsymbol{\Phi}\,\boldsymbol{z}.$$

We say *approximation* because a linear combination of $M < N$ vectors cannot describe every point in a N-space.

Rayleigh Quotient in Ritz Coordinates

We can write the Rayleigh quotient as a function of the Ritz coordinates,

$$\omega^2(\boldsymbol{z}) = \frac{\boldsymbol{z}^T \boldsymbol{\Phi}^T \boldsymbol{K}\, \boldsymbol{\Phi} \boldsymbol{z}}{\boldsymbol{z}^T \boldsymbol{\phi}^T \boldsymbol{M}\, \boldsymbol{\phi} \boldsymbol{z}} = \frac{\overline{k}(\boldsymbol{z})}{\overline{m}(\boldsymbol{z})},$$

but this is not an explicit function for any modal frequency...

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but this is not an explicit function for any modal frequency...

On the other hand, we have seen that frequency estimates are always greater than true frequencies, so our best estimates are the the local minima of $\omega^2(\boldsymbol{z})$, or the points where all the derivatives of $\omega^2(\boldsymbol{z})$ with respect to z_i are zero:

$$\frac{\partial \omega^2(\boldsymbol{z})}{\partial z_j} = \frac{\overline{m}(\boldsymbol{z}) \dfrac{\partial \overline{k}(\boldsymbol{z})}{\partial z_i} - \overline{k}(\boldsymbol{z}) \dfrac{\partial \overline{m}(\boldsymbol{z})}{\partial z_i}}{(\overline{m}(\boldsymbol{z}))^2} = 0, \qquad \text{for } i = 1, \ldots, M < N$$

Rayleigh Quotient in Ritz Coordinates

Observing that

$$\overline{k}(\boldsymbol{z}) = \omega^2(\boldsymbol{z})\overline{m}(\boldsymbol{z})$$

we can substitute into and simplify the preceding equation,

$$\frac{\partial \overline{k}(\boldsymbol{z})}{\partial z_i} - \omega^2(\boldsymbol{z})\frac{\partial \overline{m}(\boldsymbol{z})}{\partial z_i} = 0, \qquad \text{for } i = 1, \ldots, M < N$$

Rayleigh Quotient in Ritz Coordinates

With the positions

$$\boldsymbol{\Phi}^T \boldsymbol{K}\, \boldsymbol{\Phi} = \overline{\boldsymbol{K}} \qquad \text{and} \qquad \boldsymbol{\Phi}^T \boldsymbol{M}\, \boldsymbol{\Phi} = \overline{\boldsymbol{M}}$$

we have

$$\overline{k}(\boldsymbol{z}) = \boldsymbol{z}^T \overline{\boldsymbol{K}} \boldsymbol{z} = \sum_r \sum_s \overline{k}_{rs} z_r z_s,$$

hence

$$\left\{ \frac{\partial \overline{k}(\boldsymbol{z})}{\partial z_i} \right\} = \left\{ \sum_s \overline{k}_{is} z_s + \sum_r \overline{k}_{ri} z_r \right\}.$$

Due to symmetry, $\overline{k}_{ri} = \overline{k}_{ir}$ and consequently

$$\left\{ \frac{\partial \overline{k}(\boldsymbol{z})}{\partial z_i} \right\} = \left\{ 2 \sum_s \overline{k}_{is} z_s \right\} = 2 \overline{\boldsymbol{K}} \boldsymbol{z}.$$

Analogously

$$\left\{ \frac{\partial \overline{m}(\boldsymbol{z})}{\partial z_i} \right\} = 2 \overline{\boldsymbol{M}} \boldsymbol{z}.$$

Reduced Eigenproblem

Substituting these results in $\frac{\partial \overline{k}(\boldsymbol{z})}{\partial z_i} - \omega^2(\boldsymbol{z})\frac{\partial \overline{m}(\boldsymbol{z})}{\partial z_i} = 0$ we can write a *new eigenvector problem*, in the M *DOF* Ritz coordinates space, with reduced $M \times M$ matrices:

$$\overline{\boldsymbol{K}}\,\boldsymbol{z} - \omega^2 \overline{\boldsymbol{M}}\,\boldsymbol{z} = \boldsymbol{0}.$$

Modal Superposition?

After solving the reduced eigenproblem, we have a set of M eigenvalues $\overline{\omega}_i^2$ and a corresponding set of M eigenvectors $\overline{\boldsymbol{z}}_i$. What is the relation between these results and the eigenpairs of the original problem?
The $\overline{\omega}_i^2$ clearly are approximations from above to the real eigenvalues, and if we write $\overline{\boldsymbol{\psi}}_i = \boldsymbol{\Phi}\overline{\boldsymbol{z}}_i$ we see that, being

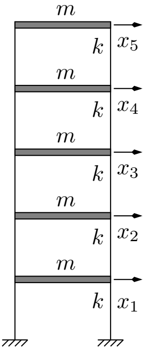
$$\overline{\boldsymbol{\psi}}_i^T \boldsymbol{M} \overline{\boldsymbol{\psi}}_j = \overline{\boldsymbol{z}}_i^T \underbrace{\boldsymbol{\Phi}^T \boldsymbol{M} \boldsymbol{\Phi}}_{\overline{\boldsymbol{M}}} \overline{\boldsymbol{z}}_j = \overline{M}_i \delta_{ij},$$

the approximated eigenvectors $\overline{\boldsymbol{\psi}}_i$ are orthogonal with respect to the structural matrices and can be used in ordinary modal superposition techniques.

A Last Question

One last question: how many $\overline{\omega}_i^2$ and $\overline{\boldsymbol{\psi}}_i$ are *effective* approximations to the true eigenpairs? Experience tells that an effective approximation is to be expected for the first $M/2$ eigenthings.

RR Example



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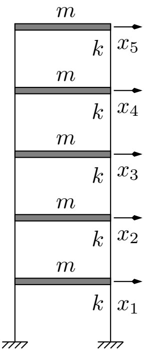

The structural matrices

$$\boldsymbol{K} = k \begin{bmatrix} +2 & -1 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{bmatrix} \quad \boldsymbol{M} = m \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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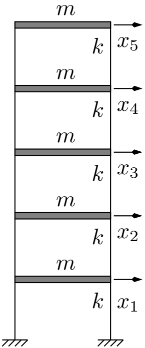

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The Ritz base vectors and the reduced matrices,

$$\boldsymbol{\Phi} = \begin{bmatrix} 0.2 & -0.5 \\ 0.4 & -1.0 \\ 0.6 & -0.5 \\ 0.8 & +0.0 \\ 1.0 & 1.0 \end{bmatrix} \quad \bar{\boldsymbol{K}} = k \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 2.0 \end{bmatrix} \quad \bar{\boldsymbol{M}} = m \begin{bmatrix} 2.2 & 0.2 \\ 0.2 & 2.5 \end{bmatrix}$$

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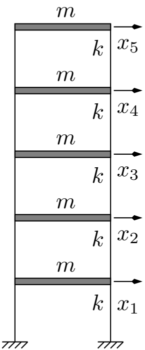

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Red. eigenproblem ($\rho = \omega^2\, m/k$): $\begin{bmatrix} 2 - 22\rho & 2 - 2\rho \\ 2 - 2\rho & 20 - 25\rho \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

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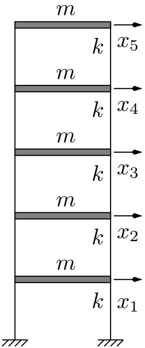

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The roots are $\rho_1 = 0.0824$, $\rho_2 = 0.800$, the frequencies are $\omega_1 = 0.287\sqrt{k/m}$ $[\,= 0.285]$, $\omega_2 = 0.850\sqrt{k/m}$ $[\,= 0.831]$, while the k/m normalized exact eigenvalues are $[0.08101405, 0.69027853]$.

The first eigenvalue is estimated with good approximation.

Rayleigh-Ritz Example

The Ritz coordinates eigenvector matrix is $\boldsymbol{Z} = \begin{bmatrix} 1.329 & 0.03170 \\ -0.1360 & 1.240 \end{bmatrix}$.

The *RR* eigenvector matrix, $\boldsymbol{\Phi}$ and the exact one, $\boldsymbol{\Psi}$:

$$\boldsymbol{\Phi} = \begin{bmatrix} +0.3338 & -0.6135 \\ +0.6676 & -1.2270 \\ +0.8654 & -0.6008 \\ +1.0632 & +0.0254 \\ +1.1932 & +1.2713 \end{bmatrix}, \qquad \boldsymbol{\Psi} = \begin{bmatrix} +0.3338 & -0.8398 \\ +0.6405 & -1.0999 \\ +0.8954 & -0.6008 \\ +1.0779 & +0.3131 \\ +1.1932 & +1.0108 \end{bmatrix}.$$

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It may be interesting to use $\hat{\boldsymbol{\Phi}} = \boldsymbol{K}^{-1}\boldsymbol{M}\,\boldsymbol{\Phi}$ as a new Ritz base to get a new estimate of the Ritz and of the structural eigenpairs.

Introduction to Subspace Iteration

Rayleigh-Ritz gives good estimates for $p \approx M/2$ modes, due also to the arbitrariness in the choice of the Ritz reduced base $\mathbf{\Phi}$.
Having to solve a $M = 2p$ order problem to find p eigenvalues is very costly, as the operation count is $\propto O(M^3)$.

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The *Subspace Iteration* procedure is a variant of the Matrix Iteration procedure, where we apply the same idea, to use the response to inertial loading in the next step, not to a single vector but to a set of different vectors at once.

Statement of the procedure

The first M eigenvalue equations can be written in matrix algebra, in terms of an $N \times M$ matrix of eigenvectors $\boldsymbol{\Phi}$ and an $M \times M$ diagonal matrix $\boldsymbol{\Lambda}$ that collects the eigenvalues

$$\underset{N\times N}{\boldsymbol{K}}\ \underset{N\times M}{\boldsymbol{\Phi}} = \underset{N\times N}{\boldsymbol{M}}\ \underset{N\times M}{\boldsymbol{\Phi}}\ \underset{M\times M}{\boldsymbol{\Lambda}}$$

Using again the hat notation for the unnormalized iterate, from the previous equation we can write

$$\boldsymbol{K}\hat{\boldsymbol{\Phi}}_1 = \boldsymbol{M}\boldsymbol{\Phi}_0$$

where $\boldsymbol{\Phi}_0$ is the matrix, $N \times M$, of the zero order trial vectors, and $\hat{\boldsymbol{\Phi}}_1$ is the matrix of the non-normalized first order trial vectors.

Orthonormalization

To proceed with iterations,

1. the trial vectors in $\hat{\mathbf{\Phi}}_{n+1}$ must be orthogonalized, so that each trial vector converges to a *different* eigenvector instead of collapsing to the first eigenvector,
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These operations can be performed in different ways (e.g., ortho-normalization by Gram-Schmidt process).
Another possibility to do both tasks at once is to solve a Rayleigh-Ritz eigenvalue problem, defined in the Ritz base constituted by the vectors in $\hat{\mathbf{\Phi}}_{n+1}$.

Associated Eigenvalue Problem

Developing the procedure for $n = 0$, with the generalized matrices

$$\boldsymbol{K}_1^\star = \hat{\boldsymbol{\Phi}}_1{}^T \boldsymbol{K} \hat{\boldsymbol{\Phi}}_1$$

and

$$\boldsymbol{M}_1^\star = \hat{\boldsymbol{\Phi}}_1{}^T \boldsymbol{M} \hat{\boldsymbol{\Phi}}_1$$

the Rayleigh-Ritz eigenvalue problem associated with the orthonormalisation of $\hat{\boldsymbol{\Phi}}_1$ is

$$\boldsymbol{K}_1^\star \hat{\boldsymbol{Z}}_1 = \boldsymbol{M}_1^\star \hat{\boldsymbol{Z}}_1 \boldsymbol{\Omega}_1^2.$$

After solving for the Ritz coordinates mode shapes, $\hat{\boldsymbol{Z}}_1$ and the frequencies $\boldsymbol{\Omega}_1^2$, using any suitable procedure, it is usually convenient to normalize the shapes, so that $\hat{\boldsymbol{Z}}_1{}^T \boldsymbol{M}_1^\star \hat{\boldsymbol{Z}}_1 = \boldsymbol{I}$. The ortho-normalized set of trial vectors at the end of the iteration is then written as

$$\boldsymbol{\Phi}_1 = \hat{\boldsymbol{\Phi}}_1 \hat{\boldsymbol{Z}}_1.$$

The entire process can be repeated for $n = 1$, then $n = 2$, $n = \dots$ until the eigenvalues converge within a prescribed tolerance.

Convergence

In principle, the procedure will converge to all the M lower eigenvalues and eigenvectors of the structural problem, but it was found that the subspace iteration method converges faster to the lower p eigenpairs, those required for dynamic analysis, if there is some additional trial vector; on the other hand, too many additional trial vectors slow down the computation without ulterior benefits.

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Experience has shown that the optimal total number M of trial vectors is the minimum of $2p$ and $p + 8$.
The subspace iteration method makes it possible to compute simultaneosly a set of eigenpairs within any required level of approximation, and is the preferred method to compute the eigenpairs of a complex dynamic system.