SDoF Linear Oscillator

Response to Harmonic Loading

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Outline of parts 1 and 2

Outline of parts 3 and 4

Vibration Isolation

Evaluation of damping

Part I

Response of an Undamped Oscillator to Harmonic Load

The Equation of Motion

The *SDOF* equation of motion for a harmonic loading is:

$$m\,\ddot{x} + k\,x = p_0\ \sin\omega t.$$

The solution can be written, using superposition, as the free vibration solution plus a *particular solution*, $\xi = \xi(t)$

$$x(t) = A\sin\omega_n t + B\cos\omega_n t + \xi(t)$$

where $\xi(t)$ satisfies the equation of motion,

$$m\,\ddot{\xi} + k\,\xi = p_0\sin\omega t.$$

A particular solution can be written in terms of a harmonic function with the same circular frequency of the excitation, ω,

$$\xi(t) = C\ \sin\omega t$$

whose second time derivative is

$$\ddot{\xi}(t) = -\omega^2\, C\ \sin\omega t.$$

Substituting x and its derivative with ξ and simplifying the time dependency we get

$$C\ (k - \omega^2 m) = p_0,$$

collecting k and introducing the *frequency ratio* $\beta = \omega/\omega_n$

$$C\, k(1 - \omega^2 m/k) = C\, k(1 - \omega^2/\omega_\mathrm{n}^2) = C\, k\ (1 - \beta^2) = p_0.$$

The Particular Integral

Starting from our last equation, $C\,(k-\omega^2 m) = C\,k\,(1-\beta^2) = p_0$, and solving for C we get

$$C = \frac{p_0}{k-\omega^2 m} = \frac{p_0}{k}\,\frac{1}{1-\beta^2}.$$

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We can now write the particular solution, with the dependencies on β singled out in the second factor:

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$$\xi(t) = \frac{p_0}{k}\,\frac{1}{1-\beta^2}\,\sin\omega t.$$

The general integral for $p(t) = p_0 \sin\omega t$ is hence

$$x(t) = A\sin\omega_\text{n}t + B\cos\omega_\text{n}t + \frac{p_0}{k}\,\frac{1}{1-\beta^2}\,\sin\omega t.$$

Response Ratio and Dynamic Amplification Factor

Introducing the *static deformation*, $\Delta_{\text{st}} = p_0/k$, and the *Response Ratio*, $R(t;\ \beta)$ the particular integral is

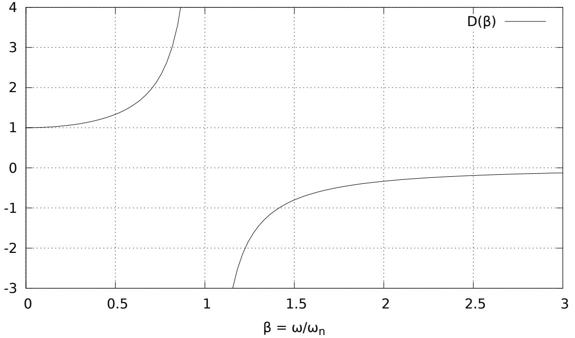
$$\xi(t) = \Delta_{\text{st}}\, R(t;\ \beta).$$

The Response Ratio is eventually expressed in terms of the *dynamic amplification factor* $D(\beta) = (1-\beta^2)^{-1}$ as follows:

$$R(t;\ \beta) = \frac{1}{1-\beta^2}\,\sin\omega t = D(\beta)\,\sin\omega t.$$

The dependency of D on β is examined in the next slide.

Dynamic Amplification Factor, the plot



$D(\beta)$ is stationary and almost equal to 1 when $\omega << \omega_n$ (this is a *quasi*-static behaviour), it grows out of bound when $\beta \Rightarrow 1$ (resonance), it is negative for $\beta > 1$ and goes to 0 when $\omega >> \omega_n$ (high-frequency loading).

Response from Rest Conditions

Starting from rest conditions means that $x(0) = 0$ and $\dot{x}(0) = 0$.

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$$x(t) = A\ \sin\omega_\mathrm{n} t + B\ \cos\omega_\mathrm{n} t + \Delta_\mathrm{st}\, D(\beta)\ \sin\omega t,$$
$$x(0) = A \times 0 + B \times 1 + \Delta_\mathrm{st}\, D(\beta) \times 0 = B = 0.$$

Response from Rest Conditions

Starting from rest conditions means that $x(0)=0$ and $\dot x(0)=0$. Let's start with $x(t)$, then evaluate $x(0)$ and finally equate this last expression to 0:

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$$x(0) = A\times 0 + B\times 1 + \Delta_\text{st}\,D(\beta)\times 0 = B = 0.$$

We do as above for the velocity:

$$\dot x(t) = \omega_\text{n}\ (A\ \cos\omega_\text{n}t - B\ \sin\omega_\text{n}t) + \Delta_\text{st}\,D(\beta)\,\omega\cos\omega t,$$
$$\dot x(0) = \omega_\text{n}\,A + \omega\,\Delta_\text{st}\,D(\beta) = 0 \Rightarrow$$
$$\Rightarrow A = -\Delta_\text{st}\,\frac{\omega}{\omega_\text{n}}D(\beta) = -\Delta_\text{st}\,\beta D(\beta)$$

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$$\dot{x}(t) = \omega_\text{n} (A \cos\omega_\text{n} t - B \sin\omega_\text{n} t) + \Delta_\text{st} D(\beta)\, \omega \cos\omega t,$$
$$\dot{x}(0) = \omega_\text{n} A + \omega\, \Delta_\text{st} D(\beta) = 0 \Rightarrow$$
$$\Rightarrow A = -\Delta_\text{st} \frac{\omega}{\omega_\text{n}} D(\beta) = -\Delta_\text{st}\, \beta D(\beta)$$

Substituting A and B in $x(t)$ above, collecting Δ_st and $D(\beta)$ we have, for $p(t) = p_0 \sin\omega t$, the response from rest:

$$x(t) = \Delta_\text{st} D(\beta) (\sin\omega t - \beta \sin\omega_\text{n} t).$$

Response from Rest Conditions, cont.

Is it different when the load is $p(t) = p_0 \cos \omega t$?

You can easily show that, similar to the previous case,

$$x(t) = x(t) = A \sin \omega_{\mathrm{n}} t + B \cos \omega_{\mathrm{n}} t + \Delta_{\mathrm{st}} D(\beta) \cos \omega t$$

and, for a system starting from rest, the initial conditions are

$$x(0) = B + \Delta_{\mathrm{st}} D(\beta) = 0$$
$$\dot{x}(0) = A = 0$$

giving $A = 0$, $B = -\Delta_{\mathrm{st}} D(\beta)$ that substituted in the general integral lead to

$$x(t) = \Delta_{\mathrm{st}} D(\beta) (\cos \omega t - \cos \omega_{\mathrm{n}} t).$$

Resonant Response from Rest Conditions

We have seen that the response to harmonic loading with zero initial conditions is

$$x(t;\beta) = \Delta_{\text{st}} \frac{\sin \omega t - \beta \, \sin \omega_{\text{n}} t}{1 - \beta^2}$$

and we know that for $\omega = \omega_n$ (i.e., $\beta = 1$) the dynamic amplification factor is infinite, but what is really happening when we have the so-called *resonant response*?

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> *The response will reach (theoretically...) an infinite amplitude but only after an infinite time, because the rate at which we can introduce energy into the system is obviously limited.*

Resonant Response from Rest Conditions

$$\frac{x(t;\beta)}{\Delta_{\mathrm{st}}} = \frac{\sin\beta\omega_{\mathrm{n}}t - \beta\sin\omega_{\mathrm{n}}t}{1-\beta^2}.$$

Resonant Response from Rest Conditions

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To determine the resonant response we will use the rule of *de l'Hôpital* that states that, in the limit, the value of a $0/0$ expression equals the ratio of the derivatives of the numerator and the denominator with respect to the free parameter, here β.

First, we substitute $\beta\omega_n$ for ω, next we compute the two derivatives and finally we substitute ω_n by ω (that can be done because $\beta = 1$):

Resonant Response from Rest Conditions, cont.

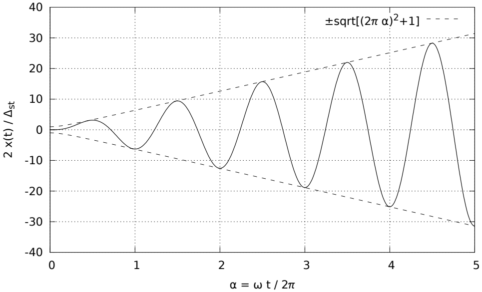
First, we substitute $\beta\omega_\text{n}$ for ω, next we compute the two derivatives and finally we substitute ω_n by ω (that can be done because $\beta = 1$):

$$\lim_{\beta\to 1} x(t;\beta) = \lim_{\beta\to 1} \Delta_\text{st} \frac{\partial(\sin\beta\omega_\text{n}t - \beta\sin\omega_\text{n}t)/\partial\beta}{\partial(1-\beta^2)/\partial\beta}$$
$$= \frac{\Delta_\text{st}}{2}\,(\sin\omega t - \omega t\cos\omega t)\,.$$

As you can see, there is a term in quadrature with the loading, whose amplitude grows linearly and without bounds.

Resonant Response, the plot

$$\frac{2}{\Delta_{\text{st}}}x(t) = \sin\omega t - \omega t\cos\omega t.$$



The amplitude $\mathcal{A}$ of the normalized envelope is

$$\mathcal{A} = \sqrt{1 + (\omega t)^2}$$

because the two components of the response are in *quadrature*.

When $\omega t \to \infty$ we have that $\mathcal{A} \to \omega t$.

Homework

Derive the expression for the resonant response when $p(t) = p_0 \cos \omega t$,

$$\lim_{\beta\to 1} x(t) = \lim_{\beta\to 1} \left(\Delta_{\mathrm{st}}\, D(\beta)\, (\cos \omega t - \cos \omega_{\mathrm{n}} t) \right).$$

Part II

Response of the Damped Oscillator to Harmonic Loading

The *SDOF* equation of motion for a harmonic loading is:

$$m\,\ddot{x} + c\,\dot{x} + k\,x = p_0\,\sin\omega t.$$

A particular solution to this equation is a harmonic function not in phase with the input: $x(t) = G\sin(\omega t - \theta)$;

The Equation of Motion for a Damped Oscillator

The *SDOF* equation of motion for a harmonic loading is:

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$$\xi(t) = G_1\ \sin\omega t + G_2\ \cos\omega t,$$

where we have simply a different formulation, no more in terms of amplitude and phase but in terms of the amplitudes of two harmonics in quadrature, as in any case the particular integral depends on two free parameters.

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The Particular Integral

Substituting $x(t)$ with $\xi(t)$, dividing by m it is

$$\ddot{\xi}(t) + 2\zeta\omega_\mathrm{n}\dot{\xi}(t) + \omega_\mathrm{n}^2\xi(t) = \frac{p_0}{k}\frac{k}{m}\,\sin\omega t,$$

Substituting the most general expressions for the particular integral and its time derivatives

$$\begin{aligned}
\xi(t) &= G_1\,\sin\omega t + G_2\,\cos\omega t,\\
\dot{\xi}(t) &= \omega\,(G_1\,cos\omega t - G_2\,\sin\omega t),\\
\ddot{\xi}(t) &= -\omega^2\,(G_1\,\sin\omega t + G_2\,\cos\omega t).
\end{aligned}$$

in the above equation it is

$$\begin{aligned}
&-\omega^2\,(G_1\,\sin\omega t + G_2\,\cos\omega t) + 2\zeta\omega_\mathrm{n}\omega\,(G_1\,cos\omega t - G_2\,\sin\omega t) +\\
&+\omega_\mathrm{n}^2(G_1\,\sin\omega t + G_2\,\cos\omega t) = \Delta_\mathrm{st}\omega_\mathrm{n}^2\sin\omega t
\end{aligned}$$

The particular integral, 2

Dividing our last equation by ω_n^2 and collecting $\sin\omega t$ and $\cos\omega t$ we obtain

$$\left(G_1(1-\beta^2)-G_2 2\beta\zeta\right)\sin\omega t+\left(G_1 2\beta\zeta+G_2(1-\beta^2)\right)\cos\omega t=\Delta_\mathrm{st}\,\sin\omega t+0\,\cos\omega t.$$

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Equating the coefficients of the sin and the cosine on both sides, we obtain a linear system of two equations in G_1 and G_2:

$$\begin{cases}G_1(1-\beta^2)-G_2 2\zeta\beta=\Delta_\text{st}.\\G_1 2\zeta\beta+G_2(1-\beta^2)=0.\end{cases}\rightarrow\begin{bmatrix}(1-\beta^2) & -2\zeta\beta\\2\zeta\beta & (1-\beta^2)\end{bmatrix}\begin{Bmatrix}G_1\\G_2\end{Bmatrix}=\begin{Bmatrix}\Delta_\text{st}\\0\end{Bmatrix}.$$

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$$\begin{cases} G_1(1-\beta^2) - G_2 2\zeta\beta = \Delta_\text{st}. \\ G_1 2\zeta\beta + G_2(1-\beta^2) = 0. \end{cases} \rightarrow \begin{bmatrix} (1-\beta^2) & -2\zeta\beta \\ 2\zeta\beta & (1-\beta^2) \end{bmatrix} \begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_\text{st} \\ 0 \end{Bmatrix}.$$

The determinant of the linear system is

$$\det = (1-\beta^2)^2 + (2\zeta\beta)^2,$$

the solution of the linear system is

$$G_1 = +\Delta_\text{st}\frac{(1-\beta^2)}{\det}, \qquad G_2 = -\Delta_\text{st}\frac{2\zeta\beta}{\det}$$

and the particular integral is

$$\xi(t) = \frac{\Delta_\text{st}}{\det}\left((1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t\right).$$

The Particular Integral, 3

Substituting G_1 and G_2 in our expression of the particular integral it is

$$\xi(t) = \frac{\Delta_{\mathrm{st}}}{\det}\left((1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t\right).$$

The general integral for $p(t) = p_0 \sin\omega t$ is hence

$$x(t) = \exp(-\zeta\omega_{\mathrm{n}}t)\left(A\, sin\omega_{\mathrm{D}}t + B\,\cos\omega_{\mathrm{D}}t\right) + \Delta_{\mathrm{st}}\frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det}$$

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For standard initial conditions we have

$$B = x_0 - G_2, \qquad A = \frac{\dot{x}_0 + \zeta\omega_n B - \omega\, G_1}{\omega_D}.$$

The Particular Integral, 4

For a generic harmonic load

$$p(t) = p_{\text{sin}}\ \sin\omega t + p_{\text{cos}}\ \cos\omega t,$$

with $\Delta_{\text{sin}} = p_{\text{sin}}/k$ and $\Delta_{\text{cos}} = p_{\text{cos}}/k$ the integral of the motion is

$$x(t) = \exp(-\zeta\omega_{\text{n}} t)\,(A\,sin\omega_{\text{D}} t + B\ \cos\omega_{\text{D}} t) + \\ + \Delta_{\text{sin}} \frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\text{det}} + \\ + \Delta_{\text{cos}} \frac{(1-\beta^2)\cos\omega t + 2\beta\zeta\sin\omega t}{\text{det}}.$$

Stationary Response

Examination of the general integral

$$x(t) = \exp(-\zeta\omega_\mathrm{n} t)\,(A\, sin\omega_\mathrm{D} t + B\ \cos\omega_\mathrm{D} t) + \Delta_\mathrm{st}\frac{(1-\beta^2)\sin\omega t - 2\beta\zeta\cos\omega t}{\det}$$

shows that we have a *transient response*, that depends on the initial conditions and damps out for large values of the argument of the real exponential, and a so called *steady-state response*, corresponding to the particular integral, $x_\text{s-s}(t) \equiv \xi(t)$, that remains constant in amplitude and phase as long as the external loading is being applied.

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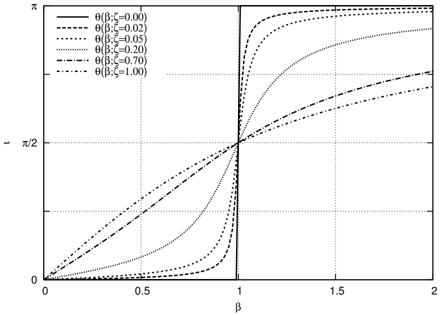
From an engineering point of view, we have a specific interest in the steady-state response, as it is the long term component of the response.

The Angle of Phase

Let's write the particular integral in terms of its amplitude and a phase difference, $G = \Delta_{\mathrm{st}} D$ and θ: $\xi(t) = \Delta_{\mathrm{st}}\, R(t;\, \beta, \zeta)$, $R = D(\beta,\zeta)\, sin\,(\omega t - \theta)$.

The phase difference θ depends on β and ζ and its expression is:

$$\theta(\beta,\zeta) = \arctan {}^{2\zeta\beta}/_{1-\beta^2}.$$



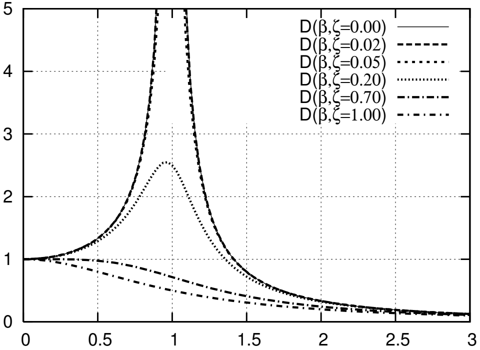
For small values of ζ $\theta(\beta,\zeta)$ has a sharp variation around $\beta = 1$ and in the case of lightly damped structures the response is approximately in phase or in opposition for, respectively, low and high frequencies of excitation.

It is worth mentioning that for $\beta = 1$ the response is always in perfect quadrature with the load, a fact that enables to detect resonant response in dynamic tests of structures.

Dynamic Magnification Ratio

The dynamic magnification factor, $D = D(\beta, \zeta)$, is the amplitude of the stationary response normalized with respect to Δ_{st}:

$$D(\beta, \zeta) = \frac{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}}{(1-\beta^2)^2 + (2\beta\zeta)^2} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}}$$



- $D(\beta, \zeta)$ has larger peak values for decreasing values of ζ,
- the approximate value of the peak, very good for a slightly damped structure, is $1/2\zeta$,
- for larger damping, peaks move toward the origin and for $\zeta = \frac{1}{\sqrt{2}}$ the peak is in the origin,
- for damping ratios $\zeta > \frac{1}{\sqrt{2}}$ we have a single maximum for $\beta = 0$.

Dynamic Magnification Ratio (2)

The location of the response peak is given by the equation

$$\frac{d\,D(\beta,\zeta)}{d\,\beta} = 0, \quad \Rightarrow \quad \beta^3 + (2\zeta^2 - 1)\,\beta = 0$$

the 3 roots are

$$\beta_i = 0, \pm\sqrt{1-2\zeta^2}.$$

We are interested in a real, positive root, so we are restricted to $0 < \zeta \le \frac{1}{\sqrt{2}}$.

In this interval, substituting $\beta = \sqrt{1-2\zeta^2}$ in the expression of the response ratio, we have

$$D_{\text{max}} = \frac{1}{2\zeta}\frac{1}{\sqrt{1-\zeta^2}} \approx \frac{1}{2\zeta} \text{ for small values of } \zeta.$$

When $\zeta = \frac{1}{\sqrt{2}}$ the equation $\beta^3 + (2\zeta^2 - 1)\,\beta = \beta^3 = 0$ has a triple root for $\beta = 0$ or, in other words, we have a very flat maximum.

> *Note that, for a relatively large damping ratio, $\zeta = 20\%$, the error of $1/2\zeta$ with respect to D_{max} is in the order of 2%.*

Harmonic Exponential Load

Consider the *EOM* for a load modulated by an exponential of imaginary argument:

$$\ddot{x} + 2\zeta\omega_\text{n}\dot{x} + \omega_\text{n}^2 x = \Delta_\text{st}\omega_\text{n}^2 \exp(i(\omega t - \phi)).$$

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The particular solution and its derivatives are

$$\begin{aligned}
\xi &= G \exp(i\omega t - i\phi),\\
\dot{\xi} &= i\omega G \exp(i\omega t - i\phi),\\
\ddot{\xi} &= -\omega^2 G \exp(i\omega t - i\phi),
\end{aligned}$$

where G is a complex constant.

Substituting, dividing by ω_n^2, removing the dependency on $\exp(i\omega t)$ and solving for G yields

$$G = \Delta_\text{st}\left[\frac{1}{(1-\beta^2)+i(2\zeta\beta)}\right] = \Delta_\text{st}\left[\frac{(1-\beta^2)-i(2\zeta\beta)}{(1-\beta^2)^2+(2\zeta\beta)^2}\right].$$

We can write

$$x_\text{s-s} = \Delta_\text{st} D(\beta,\zeta)\exp i\omega t$$

with

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It is simpler to represent the stationary response of a damped oscillator using the complex exponential representation.

Measuring Support Accelerations

We have seen that in seismic analysis the loading is proportional to the ground acceleration.
With the equation of motion valid for a harmonic support acceleration:

$$\ddot{x} + 2\zeta\beta\omega_\mathrm{n}\dot{x} + \omega_\mathrm{n}^2 x = -a_g \sin \omega t,$$

the stationary response is $\xi = \frac{m\, a_g}{k} D(\beta,\zeta)\, \sin(\omega t - \theta)$.
If the damping ratio of the oscillator is $\zeta \approxeq 0.7$, then the Dynamic Amplification $D(\beta) \approxeq 1$ for $0.0 < \beta < 0.6$.

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the stationary response is $\xi = \frac{m\,a_g}{k} D(\beta,\zeta)\,\sin(\omega t - \theta)$.

If the damping ratio of the oscillator is $\zeta \approxeq 0.7$, then the Dynamic Amplification $D(\beta) \approxeq 1$ for $0.0 < \beta < 0.6$.

Oscillator's displacements will be proportional to the accelerations of the support for applied frequencies up to about six-tenths of the natural frequency of the instrument.

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This is not the whole story, entire books have been written on the problem of *exactly* recovering the support acceleration from an accelerographic record.

Measuring Displacements

Consider now a harmonic *displacement* of the support,

$$u_g(t) = u_g \sin \omega t.$$

The support acceleration (disregarding the sign) is

$$a_g(t) = \omega^2 u_g \sin \omega t$$

the equation of motion is

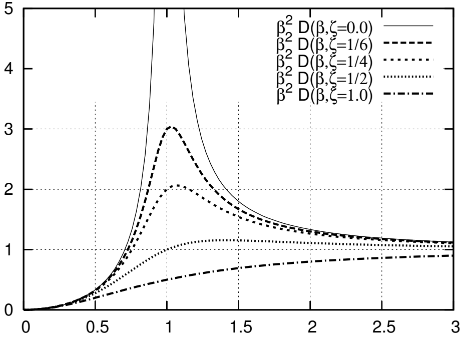
$$\ddot{x} + 2\zeta\beta\omega_\mathrm{n}\dot{x} + \omega_\mathrm{n}^2 x = -\omega^2 u_g \sin \omega t,$$

and eventually the stationary response is $\xi = u_g\,\beta^2\,D(\beta,\zeta)\,\sin(\omega t - \theta)$.

Measuring Displacements

Let's see a graph of the dynamic amplification factor derived above.

We see that the displacement of the instrument is approximately equal to the support displacement for all the excitation frequencies greater than the natural frequency of the instrument, for a damping ratio $\zeta \approxeq .5$.



It is possible to measure the support displacement measuring the deflection of the oscillator, within an almost constant scale factor, for excitation frequencies larger than ω_n.

Part III

Vibration Isolation

Vibration Isolation

Vibration isolation is a subject too broad to be treated in detail, we'll present the basic principles involved in two problems,

1. prevention of harmful vibrations in supporting structures due to oscillatory forces produced by operating equipment,
2. prevention of harmful vibrations in sensitive instruments due to vibrations of their supporting structures.

Force Isolation

Consider a rotating machine that produces an oscillatory force $p_0 \sin \omega t$ due to unbalance in its rotating part, that has a total mass m and is mounted on a spring-damper support.
Its steady-state relative displacement is given by

$$x_{\text{s-s}} = \frac{p_0}{k} D \sin(\omega t - \theta).$$

This result depend on the assumption that the supporting structure deflections are negligible respect to the relative system motion.
The steady-state spring and damper forces are

$$f_S = k\, x_{\text{ss}} = p_0\, D \sin(\omega t - \theta),$$

$$f_D = c\, \dot{x}_{\text{ss}} = \frac{c p_0\, D\, \omega}{k} \cos(\omega t - \theta) = 2\, \zeta\, \beta\, p_0\, D \cos(\omega t - \theta).$$

Transmitted force

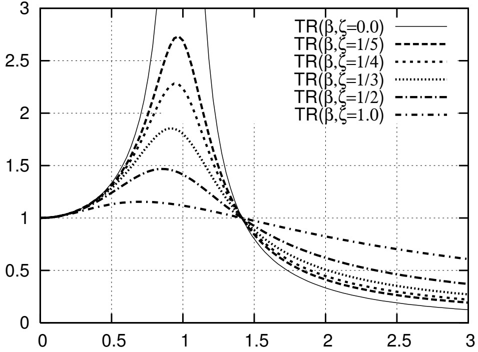
The spring and damper forces are in quadrature, so the amplitude of the steady-state reaction force is given by

$$f_{\text{max}} = p_0\, D \sqrt{1 + (2\zeta\beta)^2} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}.$$

Transmitted force

The ratio of the maximum transmitted force to the amplitude of the applied force is the *transmissibility ratio* (TR),

$$\text{TR} = \frac{f_{\text{max}}}{p_0} = \frac{\sqrt{1+(2\zeta\beta)^2}}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}.$$



1. For $\beta < \sqrt{2}$, TR ≥ 1, the transmitted force is not reduced.
2. For $\beta > \sqrt{2}$, TR < 1, note that for the same β TR is larger for larger values of ζ.

Displacement Isolation

Dual to force transmission there is the problem of the steady-state total displacements of a mass m, supported by a suspension system (i.e., spring+damper) and subjected to a harmonic motion of its base.

Let's write the base motion using the exponential notation, $u_g(t) = u_{g_0} \exp i\omega t$. The apparent force is $p_{\text{eff}} = m\omega^2 u_{g_o} \exp i\omega t$ and the steady state relative displacement is $x_{\text{ss}} = u_{g_0}\, \beta^2 D \exp i\omega t$. The mass total displacement is given by

$$\begin{aligned} x_{\text{tot}} = x_{\text{s-s}} + u_g(t) &= u_{g_0} \left(\frac{\beta^2}{(1-\beta^2) + 2\, i\, \zeta \beta} + 1 \right) \exp i\omega t \\ &= u_{g_0} (1 + 2i\zeta\beta) \frac{1}{(1-\beta^2) + 2\, i\, \zeta\beta} \exp i\omega t \\ &= u_{g_0} \sqrt{1 + (2\zeta\beta)^2}\, D \exp i\,(\omega t - \varphi). \end{aligned}$$

If we define the transmissibility ratio TR as the ratio of the maximum total response to the support displacement amplitude, we find that, as in the previous case,

$$\text{TR} = D\sqrt{1 + (2\zeta\beta)^2}.$$

Isolation Effectiveness

Define the isolation effectiveness,

$$\mathrm{IE} = 1 - \mathrm{TR},$$

IE=1 means complete isolation, i.e., $\beta = \infty$, while IE=0 is no isolation, and takes place for $\beta = \sqrt{2}$.
As effective isolation requires low damping, we can approximate $\mathrm{TR} \approxeq 1/(\beta^2 - 1)$, in which case we have $\mathrm{IE} = (\beta^2 - 2)/(\beta^2 - 1)$. Solving for β^2, we have $\beta^2 = (2 - \mathrm{IE})/(1 - \mathrm{IE})$, but

$$\beta^2 = \omega^2/\omega_\mathrm{n}^2 = \omega^2\,(m/k) = \omega^2\,(W/gk) = \omega^2\,(\Delta_\mathrm{st}/g)$$

where W is the weight of the mass and Δ_st is the static deflection under self weight. Finally, from $\omega = 2\pi f$ we have

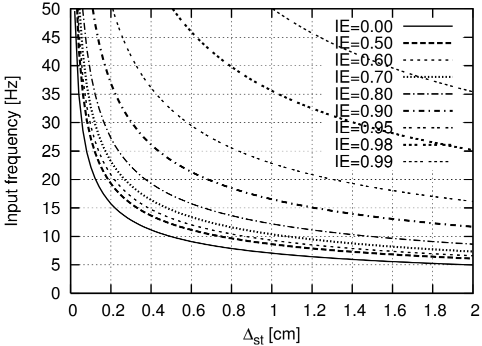
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_\mathrm{st}}\frac{2 - \mathrm{IE}}{1 - \mathrm{IE}}}$$

Isolation Effectiveness (2)

The strange looking

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_{st}}\frac{2-\text{IE}}{1-\text{IE}}}$$

can be plotted f vs Δ_{st} for different values of IE, obtaining a design chart.



Knowing the frequency of excitation and the required level of vibration isolation efficiency (IE), one can determine the minimum static deflection (proportional to the spring flexibility) required to achieve the required IE. It is apparent that any isolation system must be very flexible to be effective.

Part IV

Evaluation of Viscous Damping Ratio

Evaluation of damping

The mass and stiffness of physical systems of interest are usually evaluated easily, but this is not feasible for damping, as the energy is dissipated by different mechanisms, some one not fully understood… it is even possible that dissipation cannot be described in term of viscous-damping, But it generally is possible to measure an equivalent viscous-damping ratio by experimental methods:

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- free-vibration decay method,
- resonant amplification method,
- half-power (bandwidth) method,
- resonance cyclic energy loss method.

Free vibration decay

We already have discussed the free-vibration decay method,

$$\zeta = \frac{\delta_s}{2\pi\, s\,(\omega_{\mathrm{n}}/\omega_D)} = \frac{\delta_s}{2s\pi}\sqrt{1-\zeta^2}$$

with $\delta_s = \ln \frac{x_r}{x_{r+s}}$, *logarithmic decrement*. The method is simple and its requirements are minimal, but some care must be taken in the interpretation of free-vibration tests, because the damping ratio decreases with decreasing amplitudes of the response, meaning that for a very small amplitude of the motion the effective values of the damping can be underestimated.

Resonant amplification

This method assumes that it is possible to measure the stiffness of the structure, and that damping is small. The experimenter (a) measures the steady-state response x_{ss} of a SDOF system under a harmonic loading for a number of different excitation frequencies (eventually using a smaller frequency step when close to the resonance), (b) finds the maximum value $D_{\max} = {}^{\max\{x_{ss}\}}/_{\Delta_{st}}$ of the dynamic magnification factor, (c) uses the approximate expression (good for small ζ) $D_{\max} = {}^{1}/_{2\zeta}$ to write

$$D_{\max} = \frac{1}{2\zeta} = \frac{\max\{x_{ss}\}}{\Delta_{st}}$$

and finally (d) has

$$\zeta = \frac{\Delta_{st}}{2\max\{x_{ss}\}}.$$

The most problematic aspect here is getting a good estimate of Δ_{st}, if the results of a static test aren't available.

Half Power

The non dimensional frequencies where the response is $1/\sqrt{2}$ times the peak value can be computed from the equation

$$\frac{1}{\sqrt{(1-\beta^2)^2+(2\beta\zeta)^2}}=\frac{1}{\sqrt{2}}\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

squaring both sides and solving for β^2 gives

$$\beta_{1,2}^2=1-2\zeta^2\mp 2\zeta\sqrt{1-\zeta^2}$$

For small ζ we can use the binomial approximation and write

$$\beta_{1,2}=\left(1-2\zeta^2\mp 2\zeta\sqrt{1-\zeta^2}\right)^{\frac{1}{2}}\approxeq 1-\zeta^2\mp\zeta\sqrt{1-\zeta^2}$$

Half power (2)

From the approximate expressions for the difference of the half power frequency ratios,

$$\beta_2 - \beta_1 = 2\zeta\sqrt{1-\zeta^2} \approxeq 2\zeta$$

and their sum

$$\beta_2 + \beta_1 = 2(1-\zeta^2) \approxeq 2$$

we can deduce that

$$\frac{\beta_2-\beta_1}{\beta_2+\beta_1} = \frac{f_2-f_1}{f_2+f_1} \approxeq \frac{2\zeta\sqrt{1-\zeta^2}}{2(1-\zeta^2)} \approxeq \zeta, \text{ or } \zeta \approxeq \frac{f_2-f_1}{f_2+f_1}$$

where $f_1,\ f_2$ are the frequencies at which the steady state amplitudes equal $1/\sqrt{2}$ times the peak value, frequencies that can be determined from a dynamic test where detailed test data is available.

Resonance Cyclic Energy Loss

If it is possible to determine the phase of the s-s response, it is possible to measure ζ from the amplitude ρ of the resonant response.

At resonance, the deflections and accelerations are in quadrature with the excitation, so that the external force is equilibrated *only* by the viscous force, as both elastic and inertial forces are also in quadrature with the excitation.

The equation of dynamic equilibrium is hence:

$$p_0 = c\,\dot{x} = 2\zeta\omega_\mathrm{n} m\,(\omega_\mathrm{n}\rho).$$

Solving for ζ we obtain:

$$\zeta = \frac{p_0}{2m\omega_\mathrm{n}^2\rho}.$$