SDOF linear oscillator

Response to Periodic and Non-periodic Loadings

Giacomo Boffi

http://intranet.dica.polimi.it/people/boffi-giacomo

Dipartimento di Ingegneria Civile Ambientale e Territoriale
Politecnico di Milano

March 17, 2020

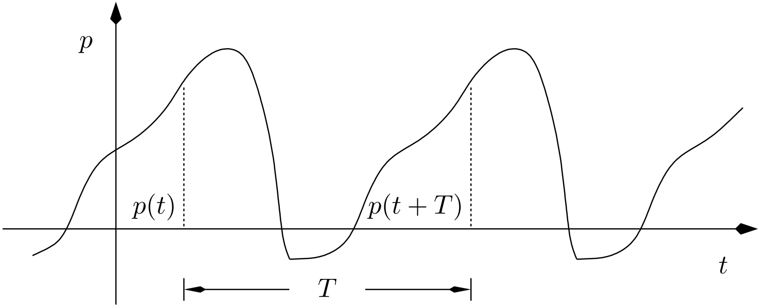
Outline

Introduction

A periodic loading is characterized by the identity

$$p(t) = p(t+T)$$

where T is the *period* of the loading, and $\omega_1 = \frac{2\pi}{T}$ is its *principal frequency*.



Introduction

Fourier theorem asserts that periodic loadings can be represented by an infinite series of harmonic functions. E.g., for an antisymmetric periodic loading of period T we have a series composed of antisymmetric harmonic functions

$$p(t) = p(-t) = \sum_{j=1}^{\infty} p_j \sin j\omega_1 t = \sum_{j=1}^{\infty} p_j \sin \omega_j t \quad (\text{with } \omega_j = j\,\tfrac{2\pi}{T}).$$

Introduction

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin\omega_j t$ is known; with $\beta_j = \omega_j/\omega_\text{n}$ the s-s response is:

$$x_{j,\text{s-s}} = \frac{p_j}{k} D(\beta_j,\zeta)\sin(\omega_j t - \theta(\beta_j,\zeta)) = a_j \cos\omega_j t + b_j \sin\omega_j t.$$

Introduction

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin \omega_j t$ is known; with $\beta_j = \omega_j/\omega_\text{n}$ the s-s response is:

$$x_{j,\text{s-s}} = \frac{p_j}{k} D(\beta_j, \zeta) \sin(\omega_j t - \theta(\beta_j, \zeta)) = a_j \cos \omega_j t + b_j \sin \omega_j t.$$

The response to an individual harmonic component can be interpreted as a term of another different Fourier series, that possibly represents the steady-state response of the dynamic system to $p(t)$.

Introduction

The steady-state response of a SDOF system for a harmonic loading $\Delta p_j(t) = p_j \sin \omega_j t$ is known; with $\beta_j = \omega_j/\omega_\text{n}$ the s-s response is:

$$x_{j,\text{s-s}} = \frac{p_j}{k} D(\beta_j, \zeta) \sin(\omega_j t - \theta(\beta_j, \zeta)) = a_j \cos \omega_j t + b_j \sin \omega_j t.$$

The response to an individual harmonic component can be interpreted as a term of another different Fourier series, that possibly represents the steady-state response of the dynamic system to $p(t)$.
It can be shown that, under very wide assumptions, the infinite series whose terms are the s-s responses to the harmonic components of $p(t)$ is indeed the Fourier series representation of the *SDOF* steady-state response to $p(t)$.

Introduction

Due to the asymptotic behaviour of $D(\beta;\zeta)$ (D goes to zero as β^{-2} for $\beta \gg 1$) it is apparent that a good approximation to the steady-state response can be obtained using a limited number of low-frequency terms.

Fourier Series

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms.

Fourier Series

Using Fourier theorem any *practical* periodic loading can be expressed as a series of harmonic loading terms.
We can represent a loading $p(t)$, of period T_p, using its Fourier series expansion

$$p(t) = \sum_{j=0}^{\infty} a_j \cos \omega_j t + \sum_{j=0}^{\infty} b_j \sin \omega_j t, \quad \omega_j = j\,\omega_1 = j\frac{2\pi}{T_\mathsf{p}}.$$

Fourier Series

Considering the orthogonality relationship over the period T_p,

$$\int_0^{T_\mathrm{p}} \cos\omega_i t \cos\omega_j t \, dt = \int_0^{T_\mathrm{p}} \sin\omega_i t \sin\omega_j t \, dt = \delta_{ij}\frac{T_\mathrm{p}}{2},$$

$$\int_0^{T_\mathrm{p}} \cos\omega_i t \sin\omega_j t \, dt = 0, \qquad \text{for} \quad i, j = 0, \ldots, \infty$$

Fourier Series

Considering the orthogonality relationship over the period T_p,

$$\int_0^{T_\mathrm{p}} \cos\omega_i t \cos\omega_j t \, dt = \int_0^{T_\mathrm{p}} \sin\omega_i t \sin\omega_j t \, dt = \delta_{ij}\frac{T_\mathrm{p}}{2},$$

$$\int_0^{T_\mathrm{p}} \cos\omega_i t \sin\omega_j t \, dt = 0, \qquad \text{for} \quad i,j = 0,\ldots,\infty$$

the amplitudes of the harmonic components are

$$a_0 = \frac{1}{T_\mathrm{p}}\int_0^{T_\mathrm{p}} p(t)\ \mathrm{d}t,\ a_j = \frac{2}{T_\mathrm{p}}\int_0^{T_\mathrm{p}} p(t)\ \cos\omega_j t\ \mathrm{d}t,$$

$$b_0 = 0\ b_j = \frac{2}{T_\mathrm{p}}\int_0^{T_\mathrm{p}} p(t)\ \sin\omega_j t\ \mathrm{d}t.$$

Note that the case $i = j = 0$ is a special case...

Fourier Coefficients

If $p(t)$ has not an analytical representation and must be measured experimentally or computed numerically, we may assume that it is possible

(a) to divide the period in N equal parts $\Delta t = T_\mathsf{p}/N$,

(b) measure or compute $p(t)$ at a discrete set of instants $t_1,\ t_2,\ \ldots, t_N$, with $t_m = m\Delta t$,

obtaining a discrete set of values $p_m,\ m = 1, \ldots, N$ (note that $p_0 = p_N$ by periodicity).

Fourier Coefficients

If $p(t)$ has not an analytical representation and must be measured experimentally or computed numerically, we may assume that it is possible

(a) to divide the period in N equal parts $\Delta t = T_\mathrm{p}/N$,

(b) measure or compute $p(t)$ at a discrete set of instants $t_1,\ t_2,\ \ldots, t_N$, with $t_m = m\Delta t$,

obtaining a discrete set of values $p_m,\ \ m = 1, \ldots, N$ (note that $p_0 = p_N$ by periodicity).

Under these assumptions the, e.g., cosine-wave amplitude coefficients can be approximated using the trapezoidal rule of integration:

$$a_j \approxeq \frac{2\Delta t}{T_\mathrm{p}} \sum_{m=1}^{N} p_m \cos\omega_j t_m = \frac{2}{N} \sum_{m=1}^{N} p_m \cos(j\omega_1 m\Delta t) = \frac{2}{N} \sum_{m=1}^{N} p_m \cos\frac{jm\,2\pi}{N}.$$

Periodicity

The coefficients of the Discrete Fourier Transform are periodic with period N.

Or, in other words, $a_{j+N} = a_j$ for $\forall j$.

Periodicity

The coefficients of the Discrete Fourier Transform are periodic with period N.

Or, in other words, $a_{j+N} = a_j$ for $\forall j$.

To prove the previous statement we'll compute a_{j+N} according to its definition:

$$a_{j+N} = \frac{2}{N}\sum_{m=1}^{N} p_m \cos\frac{2(j+N)m\pi}{N} = \frac{2}{N}\sum_{m=1}^{N} p_m \cos\frac{2(jm+Nm)\pi}{N}$$

$$= \frac{2}{N}\sum_{m=1}^{N} p_m \cos\left(\frac{2jm\pi}{N} + 2m\pi\right) = \frac{2}{N}\sum_{m=1}^{N} p_m \cos\frac{2\,jm\,\pi}{N} = a_j$$

Exponential Form

The Fourier series can also be written in terms of exponentials of imaginary argument,

$$p(t) = \sum_{j=-\infty}^{\infty} P_j \exp i\omega_j t$$

where the complex amplitude coefficients are given by

$$P_j = \frac{1}{T_\mathrm{p}} \int_0^{T_\mathrm{p}} p(t) \exp -i\omega_j t \;\mathrm{d}t, \qquad j = -\infty, \ldots, +\infty.$$

Exponential Form

The Fourier series can also be written in terms of exponentials of imaginary argument,

$$p(t) = \sum_{j=-\infty}^{\infty} P_j \exp i\omega_j t$$

where the complex amplitude coefficients are given by

$$P_j = \frac{1}{T_\mathsf{p}} \int_0^{T_\mathsf{p}} p(t) \exp -i\omega_j t \,\mathrm{d}t, \qquad j = -\infty, \ldots, +\infty.$$

For a sampled p_m we can write, using the trapezoidal integration rule and substituting $t_m = m\Delta t = m\,T_\mathsf{p}/N$, $\omega_j = j\,2\pi/T_\mathsf{p}$:

$$P_j \approxeq \frac{1}{N} \sum_{m=1}^{N} p_m \exp(-i\frac{2\pi\,j\,m}{N}).$$

For sampled input also the coefficients of the exponential series are periodic, $P_{j+N} = P_j$.

Undamped Response

We have seen that the steady-state response to the jth sine-wave harmonic can be written as

$$x_j = \frac{b_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\sin\omega_j t, \qquad \beta_j = \omega_j/\omega_\mathsf{n},$$

analogously, for the jth cosine-wave harmonic,

$$x_j = \frac{a_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\cos\omega_j t.$$

Undamped Response

We have seen that the steady-state response to the jth sine-wave harmonic can be written as

$$x_j = \frac{b_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\sin\omega_j t, \qquad \beta_j = \omega_j/\omega_\mathrm{n},$$

analogously, for the jth cosine-wave harmonic,

$$x_j = \frac{a_j}{k}\left[\frac{1}{1-\beta_j^2}\right]\cos\omega_j t.$$

Finally, we write

$$x(t) = \frac{1}{k}\left\{a_0 + \sum_{j=1}^{\infty}\left[\frac{1}{1-\beta_j^2}\right](a_j\cos\omega_j t + b_j\sin\omega_j t)\right\}.$$

Damped Response

In the case of a damped oscillator, we must substitute the steady state response for both the jth sine- and cosine-wave harmonic,

$$x(t) = \frac{a_0}{k} + \frac{1}{k}\sum_{j=1}^{\infty} \frac{+(1-\beta_j^2)\,a_j - 2\zeta\beta_j\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\cos\omega_j t + \\ + \frac{1}{k}\sum_{j=1}^{\infty} \frac{+2\zeta\beta_j\,a_j + (1-\beta_j^2)\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\sin\omega_j t.$$

Damped Response

In the case of a damped oscillator, we must substitute the steady state response for both the jth sine- and cosine-wave harmonic,

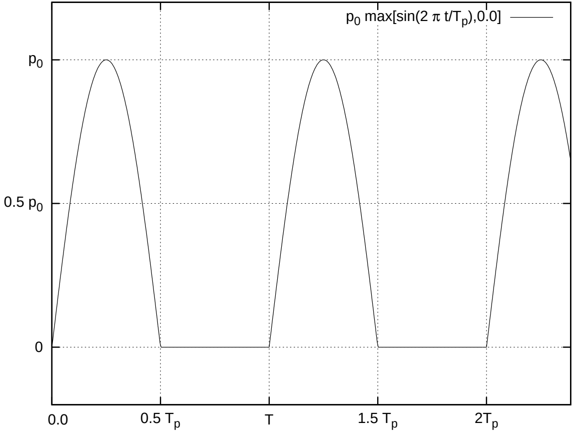
$$x(t) = \frac{a_0}{k} + \frac{1}{k}\sum_{j=1}^{\infty}\frac{+(1-\beta_j^2)\,a_j - 2\zeta\beta_j\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\cos\omega_j t + \\ + \frac{1}{k}\sum_{j=1}^{\infty}\frac{+2\zeta\beta_j\,a_j + (1-\beta_j^2)\,b_j}{(1-\beta_j^2)^2+(2\zeta\beta_j)^2}\sin\omega_j t.$$

As usual, the exponential notation is neater,

$$x(t) = \sum_{j=-\infty}^{\infty}\frac{P_j}{k}\frac{\exp i\omega_j t}{(1-\beta_j^2)+i\,(2\zeta\beta_j)}.$$

Example

As an example, consider the loading $p(t) = \max\{p_0 \sin \frac{2\pi t}{T_p}, 0\}$



Example

As an example, consider the loading $p(t) = \max\{p_0 \sin \frac{2\pi t}{T_\mathrm{p}},\ 0\}$

$$a_0 = \frac{1}{T_\mathrm{p}} \int_0^{T_\mathrm{p}/2} p_o \sin \frac{2\pi t}{T_\mathrm{p}}\, \mathrm{d}t = \frac{p_0}{\pi},$$

$$a_j = \frac{2}{T_\mathrm{p}} \int_0^{T_\mathrm{p}/2} p_o \sin \frac{2\pi t}{T_\mathrm{p}} \cos \frac{2\pi j t}{T_\mathrm{p}}\, \mathrm{d}t = \begin{cases} 0 & \text{for } j \text{ odd} \\ \frac{p_0}{\pi} \left[\frac{2}{1-j^2}\right] & \text{for } j \text{ even,} \end{cases}$$

$$b_j = \frac{2}{T_\mathrm{p}} \int_0^{T_\mathrm{p}/2} p_o \sin \frac{2\pi t}{T_\mathrm{p}} \sin \frac{2\pi j t}{T_\mathrm{p}}\, \mathrm{d}t = \begin{cases} \frac{p_0}{2} & \text{for } j = 1 \\ 0 & \text{for } n > 1. \end{cases}$$

Example cont.

Assuming $\beta_1 = 3/4$, from $p = \frac{p_0}{\pi}\left(1 + \frac{\pi}{2}\sin\omega_1 t - \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_2 t - \ldots\right)$ with the dynamic amplifiction factors

$$D_1 = \frac{1}{1-(1\frac{3}{4})^2} = \frac{16}{7}, \quad D_2 = \frac{1}{1-(2\frac{3}{4})^2} = -\frac{4}{5}, \quad D_4 = \frac{1}{1-(4\frac{3}{4})^2} = -\frac{1}{8}, \quad D_6 = \ldots$$

etc, we have

$$x(t) = \frac{p_0}{k\pi}\left(1 + \frac{8\pi}{7}\sin\omega_1 t + \frac{8}{15}\cos 2\omega_1 t + \frac{1}{60}\cos 4\omega_1 t + \ldots\right)$$

Example cont.

Assuming $\beta_1 = 3/4$, from $p = \frac{p_0}{\pi}\left(1 + \frac{\pi}{2}\sin\omega_1 t - \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_2 t - \ldots\right)$ with the dynamic amplifiction factors

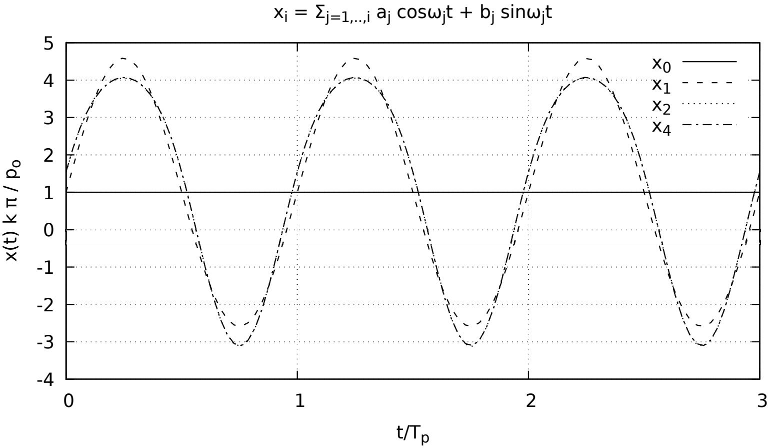
$$D_1 = \frac{1}{1-(1\frac{3}{4})^2} = \frac{16}{7}, \quad D_2 = \frac{1}{1-(2\frac{3}{4})^2} = -\frac{4}{5}, \quad D_4 = \frac{1}{1-(4\frac{3}{4})^2} = -\frac{1}{8}, \quad D_6 = \ldots$$

etc, we have

$$x(t) = \frac{p_0}{k\pi}\left(1 + \frac{8\pi}{7}\sin\omega_1 t + \frac{8}{15}\cos 2\omega_1 t + \frac{1}{60}\cos 4\omega_1 t + \ldots\right)$$

Take note, these solutions are particular solutions! If your solution has to respect given initial conditions, you must consider also the homogeneous solution.

Example cont.



The Fourier Transform

Non periodic loadings

It is possible to extend the Fourier analysis to non periodic loading. Let's start from the Fourier series representation of the load $p(t)$,

$$p(t) = \sum_{-\infty}^{+\infty} P_r \exp(i\omega_r t), \quad \omega_r = r\Delta\omega, \quad \Delta\omega = \frac{2\pi}{T_p},$$

Non periodic loadings

It is possible to extend the Fourier analysis to non periodic loading. Let's start from the Fourier series representation of the load $p(t)$,

$$p(t) = \sum_{-\infty}^{+\infty} P_r \exp(i\omega_r t), \quad \omega_r = r\Delta\omega, \quad \Delta\omega = \frac{2\pi}{T_p},$$

introducing $P(i\omega_r) = P_r T_p$ and substituting,

$$p(t) = \frac{1}{T_p} \sum_{-\infty}^{+\infty} P(i\omega_r) \exp(i\omega_r t) = \frac{\Delta\omega}{2\pi} \sum_{-\infty}^{+\infty} P(i\omega_r) \exp(i\omega_r t).$$

Non periodic loadings

It is possible to extend the Fourier analysis to non periodic loading. Let's start from the Fourier series representation of the load $p(t)$,

$$p(t) = \sum_{-\infty}^{+\infty} P_r \exp(i\omega_r t), \quad \omega_r = r\Delta\omega, \quad \Delta\omega = \frac{2\pi}{T_p},$$

introducing $P(i\omega_r) = P_r T_p$ and substituting,

$$p(t) = \frac{1}{T_p} \sum_{-\infty}^{+\infty} P(i\omega_r) \exp(i\omega_r t) = \frac{\Delta\omega}{2\pi} \sum_{-\infty}^{+\infty} P(i\omega_r) \exp(i\omega_r t).$$

Due to periodicity, we can modify the extremes of integration in the expression for the complex amplitudes,

$$P(i\omega_r) = \int_{-T_p/2}^{+T_p/2} p(t) \exp(-i\omega_r t)\, dt.$$

Non periodic loadings (2)

If the loading period is extended to infinity to represent the non-periodicity of the loading ($T_p \to \infty$) then (a) the frequency increment becomes infinitesimal ($\Delta\omega = \frac{2\pi}{T_p} \to d\omega$) and (b) the discrete frequency ω_r becomes a continuous variable, ω.
In the limit, for $T_p \to \infty$ we can then write

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(i\omega) \exp(i\omega t)\, d\omega$$

$$P(i\omega) = \int_{-\infty}^{+\infty} p(t) \exp(-i\omega t)\, dt,$$

which are known as the inverse and the direct Fourier Transforms, respectively, and are collectively known as the Fourier transform pair.

SDOF Response

In analogy to what we have seen for periodic loads, the response of a damped SDOF system can be written in terms of $H(i\omega)$, the complex frequency response function,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(i\omega)\, P(i\omega) \exp i\omega t\, dt, \quad \text{where}$$

$$H(i\omega) = \frac{1}{k}\left[\frac{1}{(1-\beta^2)+i(2\zeta\beta)}\right] = \frac{1}{k}\left[\frac{(1-\beta^2)-i(2\zeta\beta)}{(1-\beta^2)^2+(2\zeta\beta)^2}\right], \quad \beta = \frac{\omega}{\omega_n}.$$

To obtain the response *through frequency domain*, you should evaluate the above integral, but analytical integration is not always possible, and when it is possible, it is usually very difficult, implying contour integration in the complex plane (for an example, see Example **E6-3** in Clough Penzien).

The Discrete Fourier Transform

Discrete Fourier Transform

To overcome the analytical difficulties associated with the inverse Fourier transform, one can use appropriate numerical methods, leading to good approximations.

Discrete Fourier Transform

Consider a loading of finite period T_p, $p(t) = p(t + T_p)$, it is

$$P_r = \frac{1}{T_p} \int_0^{T_p} p(t) \exp(-i\omega_r t)\, dt, \quad r = -\infty, \ldots, \infty.$$

Discrete Fourier Transform

Consider a loading of finite period T_p, $p(t) = p(t + T_p)$, it is

$$P_r = \frac{1}{T_p} \int_0^{T_p} p(t) \exp(-i\omega_r t)\, dt, \quad r = -\infty, \dots, \infty.$$

Now, think that T_p has been divided in N equal intervals $\Delta t = T_p/N$ and the only thing we know about $p(t)$ is the set of N values $p_s = p(t_s) = p(s\Delta t)$.

It is still possible to compute the approximate value of every complex amplitude coefficients using the trapezoidal rule of integration.

Discrete Fourier Transform

Consider a loading of finite period T_p, $p(t) = p(t + T_p)$, it is

$$P_r = \frac{1}{T_p}\int_0^{T_p} p(t)\exp(-i\omega_r t)\,dt, \quad r = -\infty, \ldots, \infty.$$

Now, think that T_p has been divided in N equal intervals $\Delta t = T_p/N$ and the only thing we know about $p(t)$ is the set of N values $p_s = p(t_s) = p(s\Delta t)$.

It is still possible to compute the approximate value of every complex amplitude coefficients using the trapezoidal rule of integration.
Taking into account that, by peridiocity, $p_0 \equiv p_N$, it is

$$\begin{aligned}
P_r &= \frac{1}{T_p}\int_0^{T_p} p(t)\exp(-i\omega_r t)\,dt \\
&\approx \frac{1}{N\Delta t}\left(\Delta t\sum_{s=0}^{N-1} p_s\exp(-i\omega_r t_s)\right) = \frac{1}{N}\sum_{s=0}^{N-1} p_s\exp(-i\frac{2\pi rs}{N}).
\end{aligned}$$

Discrete Fourier Transform

In the last two passages we have used the following identities

$$p_N = p_0, \qquad \exp(i\omega_r t_N) = \exp(i\omega_r t_0),$$
$$\omega_r\, t_s = r\Delta\omega\, s\Delta t = rs\,\frac{2\pi}{T_p}\,\frac{T_p}{N} = \frac{2\pi\, rs}{N}.$$

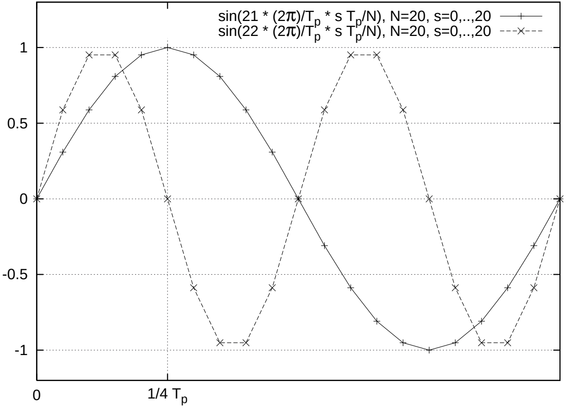
Take note that the discrete function $\exp(-i\frac{2\pi rs}{N})$, defined for integer $r,\ s$ is periodic with period N, implying that the complex amplitude coefficients are themselves periodic with period N.

$$P_{r+N} = P_r$$

Starting in the time domain with N distinct complex numbers, p_s, we have found that in the frequency domain our load is described by N distinct complex numbers, P_r, so that we can say that our function is described by the same amount of information in both domains.

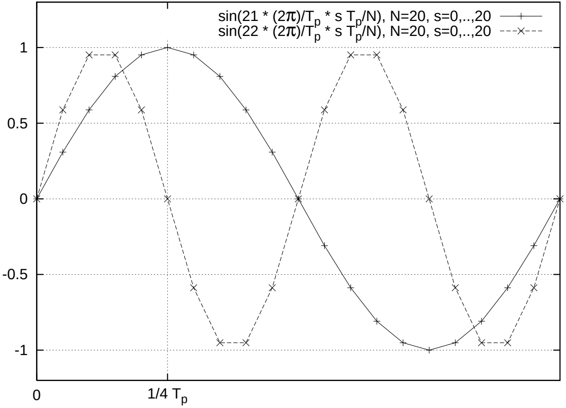
Aliasing

Only $N/2$ distinct frequencies ($\sum_0^{N-1} = \sum_{-N/2}^{+N/2}$) contribute to the load representation, what if the *frequency content* of the loading has contributions from frequencies higher than $\omega_{N/2}$? What happens is *aliasing*, i.e., the upper frequencies contributions are mapped to contributions of lesser frequency.



Aliasing

Only $N/2$ distinct frequencies $(\sum_0^{N-1} = \sum_{-N/2}^{+N/2})$ contribute to the load representation, what if the *frequency content* of the loading has contributions from frequencies higher than $\omega_{N/2}$? What happens is *aliasing*, i.e., the upper frequencies contributions are mapped to contributions of lesser frequency.



See the plot above: the contributions from the high frequency sines, *when sampled*, are indistinguishable from the contributions from lower frequency components, i.e., are *aliased* to lower frequencies!

Aliasing (2)

- The maximum frequency that can be described in the DFT is called the Nyquist frequency, $\omega_{\text{Ny}} = \frac{1}{2}\frac{2\pi}{\Delta t}$.
- It is usual in signal analysis to remove the signal's higher frequency components preprocessing the signal with a *filter* or a *digital filter*.
- It is worth noting that the *resolution* of the DFT in the frequency domain for a given sampling rate is proportional to the number of samples, i.e., to the duration of the sample.

The Fast Fourier Transform

The operation count in a DFT is in the order of N^2.
A Fast Fourier Transform is an algorithm that reduces the number of arithmetic operations needed to compute a DFT.
The first and simpler FFT algorithm is the *Decimation in Time* algorithm by Cooley and Tukey (1965).

The Fast Fourier Transform

The operation count in a DFT is in the order of N^2.
A Fast Fourier Transform is an algorithm that reduces the number of arithmetic operations needed to compute a DFT.
The first and simpler FFT algorithm is the *Decimation in Time* algorithm by Cooley and Tukey (1965).
The algorithm introduced by Cooley and Tukey is quite complex because it allows to proceed without additional memory, we will describe a different algorithm, that is based on the same principles but requires additional memory and it's rather simpler than the original one.

Decimation in Time DFT Algorithm

For simplicity, assume that N is even and split the DFT summation in two separate sums, with even and odd indices

$$X_r = \sum_{s=0}^{N-1} x_s e^{-\frac{2\pi i}{N} sr}, \qquad r = 0, \ldots, N-1$$
$$= \sum_{q=0}^{N/2-1} x_{2q} e^{-\frac{2\pi i}{N}(2q)r} + \sum_{q=0}^{N/2-1} x_{2q+1} e^{-\frac{2\pi i}{N}(2q+1)r}.$$

Collecting $e^{-\frac{2\pi i}{N}r}$ in the second term and letting $\frac{2q}{N} = \frac{q}{N/2}$, we have

Decimation in Time DFT Algorithm

For simplicity, assume that N is even and split the DFT summation in two separate sums, with even and odd indices

$$
\begin{aligned}
X_r &= \sum_{s=0}^{N-1} x_s e^{-\frac{2\pi i}{N}sr}, \qquad r = 0, \dots, N-1 \\
&= \sum_{q=0}^{N/2-1} x_{2q} e^{-\frac{2\pi i}{N}(2q)r} + \sum_{q=0}^{N/2-1} x_{2q+1} e^{-\frac{2\pi i}{N}(2q+1)r}.
\end{aligned}
$$

Collecting $e^{-\frac{2\pi i}{N}r}$ in the second term and letting $\frac{2q}{N} = \frac{q}{N/2}$, we have

$$
X_r = \sum_{q=0}^{N/2-1} x_{2q} e^{-\frac{2\pi i}{N/2}qr} + e^{-\frac{2\pi i}{N}r} \sum_{q=0}^{N/2-1} x_{2q+1} e^{-\frac{2\pi i}{N/2}qr},
$$

i.e., we have two DFT's of length $N/2$. The operations count is just $2(N/2)^2 = N^2/2$, but we have to combine these two halves in the full DFT.

Decimation in Time DFT

Say that

$$X_r = E_r + e^{-\frac{2\pi i}{N}r} O_r$$

where E_r and O_r are the even and odd half-DFT's, of which we computed only coefficients from 0 to $N/2 - 1$.
To get the full sequence we have to note that

1. the E and O DFT's are periodic with period $N/2$, and
2. $\exp(-2\pi i(r + N/2)/N) = e^{-\pi i} \exp(-2\pi i r/N) = -\exp(-2\pi i r/N)$,

so that we can write

$$X_r = \begin{cases} E_r + \exp(-2\pi i r/N) O_r & \text{if } r < N/2, \\ E_{r-N/2} - \exp(-2\pi i r/N) O_{r-N/2} & \text{if } r \geq N/2. \end{cases}$$

The algorithm that was outlined can be applied to the computation of each of the half-DFT's when $N/2$ were even, so that the operation count goes to $N^2/4$. If $N/4$ were even ...

Pseudocode for CT algorithm

```
def fft2(X, N):
  if N = 1 then
     Y = X
  else
     Y0 = fft2(X0, N/2)
     Y1 = fft2(X1, N/2)
     for k = 0 to N/2-1
       Y_k       = Y0_k + exp(2 pi i k/N) Y1_k
       Y_(k+N/2) = Y0_k - exp(2 pi i k/N) Y1_k
     endfor
  endif
return Y
```

```python
from cmath import exp, pi

def d_fft(x,n):
    ”Direct fft of x, a list of n=2**m complex values”
    return _fft(x,n,[exp(-2*pi*1j*k/n) for k in range(n/2)])

def i_fft(x,n):
    ”Inverse fft of x, a list of n=2**m complex values”
    transform = _fft(x,n,[exp(+2*pi*1j*k/n) for k in range(n/2)])]
    return [x/n for x in transform]
```

```python
def _fft(x, n, tw):
    """Decimation in Time FFT, to be called by d_fft and i_fft.
    x   is the signal to transform, a list of complex values
    n   is its length, results are undefined if n is not a power of 2
    tw  is a list of twiddle factors, precomputed by the caller

    returns a list of complex values,not normalized if inverse transform"""

    if n == 1: return x # bottom reached, DFT of a length 1 vec x is x

    # call fft with the even and the odd coefficients in x
    # the results are the so called even and odd  DFT's
    e, o = _fft(x[0::2], n/2, tw[::2]), _fft(x[1::2], n/2, tw[::2])

    # assemble the partial results:
    # 1st half of full DFT is put in even DFT, 2nd half in odd DFT
    for k in range(n/2):
        e[k], o[k] = e[k]+tw[k]*o[k], e[k]-tw[k]*o[k]

    # concatenate the two halves of the DFT and return to caller
    return e + o
```

If we strip all comments, our FFT function becomes

```python
def _fft(x, n, tw):
    if n==1:return x
    e,o=_fft(x[0::2],n/2,tw[::2]),_fft(x[1::2],n/2,tw[::2])
    for k in range(n/2):e[k],o[k]=e[k]+tw[k]*o[k],e[k]-tw[k]*o[k]
    return e+o
```

Dynamic Response (1)

To evaluate the dynamic response of a linear SDOF system in the frequency domain, use the inverse DFT,

$$x_s = \sum_{r=0}^{N-1} V_r \exp(i\frac{2\pi\, rs}{N}), \qquad s = 0, 1, \ldots, N-1$$

where $V_r = H_r\, P_r$. P_r are the discrete complex amplitude coefficients computed using the direct DFT, and H_r is the discretization of the complex frequency response function, that for viscous damping is

$$H_r = \frac{1}{k}\left[\frac{1}{(1-\beta_r^2) + i(2\zeta\beta_r)}\right] = \frac{1}{k}\left[\frac{(1-\beta_r^2) - i(2\zeta\beta_r)}{(1-\beta_r^2)^2 + (2\zeta\beta_r)^2}\right], \qquad \beta_r = \frac{\omega_r}{\omega_n}.$$

while for *hysteretic damping* it is

$$H_r = \frac{1}{k}\left[\frac{1}{(1-\beta_r^2) + i(2\zeta)}\right] = \frac{1}{k}\left[\frac{(1-\beta_r^2) - i(2\zeta)}{(1-\beta_r^2)^2 + (2\zeta)^2}\right].$$

Dynamic Response (2)

Some word of caution...
If you’re going to approach the application of the complex frequency response function without proper concern, you’re likely to be hurt.

Dynamic Response (2)

Some word of caution...
If you're going to approach the application of the complex frequency response function without proper concern, you're likely to be hurt.

Let's say $\Delta\omega = 1.0$, $N = 32$, $\omega_\mathrm{n} = 3.5$ and $r = 30$, what do you think it is the value of β_{30}?

Dynamic Response (2)

Some word of caution...
If you're going to approach the application of the complex frequency response function without proper concern, you're likely to be hurt.

Let's say $\Delta\omega = 1.0$, $N = 32$, $\omega_\mathrm{n} = 3.5$ and $r = 30$, what do you think it is the value of β_{30}?
If you are thinking $\beta_{30} = 30\,\Delta\omega/\omega_\mathrm{n} = 30/3.5 \approx 8.57$ you're wrong!

Dynamic Response (2)

Some word of caution...
If you're going to approach the application of the complex frequency response function without proper concern, you're likely to be hurt.

Let's say $\Delta\omega = 1.0$, $N = 32$, $\omega_\text{n} = 3.5$ and $r = 30$, what do you think it is the value of β_{30}?
If you are thinking $\beta_{30} = 30\,\Delta\omega/\omega_\text{n} = 30/3.5 \approx 8.57$ you're wrong!

Due to aliasing, $\omega_r = \begin{cases} r\Delta\omega & r \le N/2 \\ (r-N)\Delta\omega & r > N/2 \end{cases}$,
note that in the upper part of the DFT the coefficients correspond to negative frequencies and, staying within our example, it is $\beta_{30} = (30-32) \times 1/3.5 \approx -0.571$.
If N is even, $P_{N/2}$ is the coefficient corresponding to the Nyquist frequency, if N is odd $P_{\frac{N-1}{2}}$ corresponds to the largest positive frequency, while $P_{\frac{N+1}{2}}$ corresponds to the largest negative frequency.

Response to General Dynamic Loading

Response to a short duration load

An approximate procedure to evaluate the maximum displacement for a short impulse loading is based on the impulse-momentum relationship,

$$m\Delta\dot{x} = \int_0^{t_0} \left[p(t) - kx(t)\right]\,\mathrm{d}t.$$

When one notes that, for small t_0, the displacement is of the order of t_0^2 while the velocity is in the order of t_0, it is apparent that the kx term may be dropped from the above expression, i.e.,

$$m\Delta\dot{x} \approxeq \int_0^{t_0} p(t)\,\mathrm{d}t.$$

Response to a short duration load

Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m} \int_0^{t_0} p(t)\ \mathrm{d}t,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \approxeq 0$, one has

$$x(t - t_0) \approxeq \frac{1}{m\omega_\mathrm{n}} \int_0^{t_0} p(t)\ \mathrm{d}t\ \ \sin \omega_\mathrm{n}(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading.

Response to a short duration load

Using the previous approximation, the velocity at time t_0 is

$$\dot{x}(t_0) = \frac{1}{m} \int_0^{t_0} p(t)\ \mathrm{d}t,$$

and considering again a negligibly small displacement at the end of the loading, $x(t_0) \approxeq 0$, one has

$$x(t - t_0) \approxeq \frac{1}{m\omega_\mathsf{n}} \int_0^{t_0} p(t)\ \mathrm{d}t\ \ \sin\omega_\mathsf{n}(t - t_0).$$

Please note that the above equation is exact for an infinitesimal impulse loading.

$$\mathrm{d}x(t - \tau) = \frac{p(\tau)\ \mathrm{d}\tau}{m\omega_\mathsf{n}} \sin\omega_\mathsf{n}(t - \tau), \quad t > \tau,$$

Undamped SDOF

For an infinitesimal impulse, the impulse-momentum is exactly $p(\tau)\ \mathrm{d}\tau$ and the response is

$$\mathrm{d}x(t-\tau) = \frac{p(\tau)\ \mathrm{d}\tau}{m\omega_\mathsf{n}} \sin\omega_\mathsf{n}(t-\tau), \quad t > \tau,$$

and to evaluate the response at time t one has simply to sum all the infinitesimal contributions for $\tau < t$,

$$x(t) = \frac{1}{m\omega_\mathsf{n}} \int_0^t p(\tau)\ \sin\omega_\mathsf{n}(t-\tau)\ \mathrm{d}\tau, \quad t > 0.$$

This relation is known as the Duhamel integral, and tacitly depends on initial rest conditions for the system.

Undamped SDOF

For an infinitesimal impulse, the impulse-momentum is exactly $p(\tau)\,\mathrm{d}\tau$ and the response is

$$\mathrm{d}x(t-\tau) = \frac{p(\tau)\,\mathrm{d}\tau}{m\omega_\mathsf{n}} \sin\omega_\mathsf{n}(t-\tau), \quad t > \tau,$$

and to evaluate the response at time t one has simply to sum all the infinitesimal contributions for $\tau < t$,

$$x(t) = \frac{1}{m\omega_\mathsf{n}} \int_0^t p(\tau)\,\sin\omega_\mathsf{n}(t-\tau)\,\mathrm{d}\tau, \quad t > 0.$$

This relation is known as the Duhamel integral, and tacitly depends on initial rest conditions for the system.

Jean-Marie Constant Duhamel (Saint-Malo, 5 February 1797 — Paris, 29 April 1872)

Damped SDOF

The derivation of the equation of motion for a generic load is analogous to what we have seen for undamped SDOF, the infinitesimal contribution to the response at time t of the load at time τ is

$$\mathrm{d}x(t) = \frac{p(\tau)}{m\omega_D}\,\mathrm{d}\tau\,\sin\omega_D(t-\tau)\exp(-\zeta\omega_\mathsf{n}(t-\tau)) \quad t \geq \tau$$

and integrating all infinitesimal contributions one has

$$x(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)\,\sin\omega_D(t-\tau)\exp(-\zeta\omega_\mathsf{n}(t-\tau))\,\mathrm{d}\tau, \quad t \geq 0.$$

Evaluation of Duhamel integral, undamped

Using the trigonomtric identity

$$\sin(\omega_\mathsf{n} t - \omega_\mathsf{n}\tau) = \sin\omega_\mathsf{n} t\cos\omega_\mathsf{n}\tau - \cos\omega_\mathsf{n} t\sin\omega_\mathsf{n}\tau$$

the Duhamel integral is rewritten as

$$\begin{aligned} x(t) &= \frac{\int_0^t p(\tau)\cos\omega_\mathsf{n}\tau\ \mathrm{d}\tau}{m\omega_\mathsf{n}}\sin\omega_\mathsf{n} t - \frac{\int_0^t p(\tau)\sin\omega_\mathsf{n}\tau\ \mathrm{d}\tau}{m\omega_\mathsf{n}}\cos\omega_\mathsf{n} t \\ &= \mathcal{A}(t)\sin\omega_\mathsf{n} t - \mathcal{B}(t)\cos\omega_\mathsf{n} t \end{aligned}$$

where

$$\begin{cases} \mathcal{A}(t) = \frac{1}{m\omega_\mathsf{n}}\int_0^t p(\tau)\cos\omega_\mathsf{n}\tau\ \mathrm{d}\tau \\ \mathcal{B}(t) = \frac{1}{m\omega_\mathsf{n}}\int_0^t p(\tau)\sin\omega_\mathsf{n}\tau\ \mathrm{d}\tau \end{cases}$$

Numerical evaluation of Duhamel integral, undamped

Usual numerical procedures can be applied to the evaluation of $\mathcal{A}$ and $\mathcal{B}$, e.g., using the trapezoidal rule, one can have, with $\mathcal{A}_n = \mathcal{A}(n\Delta\tau)$, $y_n = p(n\Delta\tau)\cos(n\Delta\tau)$ and $z_n = p(n\Delta\tau)\sin(n\Delta\tau)$ we can write

$$\mathcal{A}_{n+1} = \mathcal{A}_n + \frac{\Delta\tau}{2m\omega_\mathrm{n}}\left(y_n + y_{n+1}\right),$$

$$\mathcal{B}_{n+1} = \mathcal{B}_n + \frac{\Delta\tau}{2m\omega_\mathrm{n}}\left(z_n + z_{n+1}\right).$$

Evaluation of Duhamel integral, damped

For a damped system, it can be shown that

$$x(t) = \mathcal{A}(t)\sin\omega_D t - \mathcal{B}(t)\cos\omega_D t$$

with

$$\mathcal{A}(t) = \frac{\exp -\zeta\omega_\mathsf{n} t}{m\omega_D}\int_0^t p(\tau)\exp\zeta\omega_\mathsf{n}\tau\cos\omega_D\tau\;\mathrm{d}\tau,$$

$$\mathcal{B}(t) = \frac{\exp -\zeta\omega_\mathsf{n} t}{m\omega_D}\int_0^t p(\tau)\exp\zeta\omega_\mathsf{n}\tau\sin\omega_D\tau\;\mathrm{d}\tau.$$

Numerical evaluation of Duhamel integral, damped

Numerically, using e.g. Simpson integration rule and $y_n = p(n\Delta\tau)\cos\omega_D\tau$,

$$\begin{aligned}\mathcal{A}_{n+2} = \mathcal{A}_n \exp(-2\zeta\omega_\text{n}\Delta\tau) + &\\ \frac{\Delta\tau}{3m\omega_D}\left[y_n \exp(-2\zeta\omega_\text{n}\Delta\tau) + 4y_{n+1}\exp(-\zeta\omega_\text{n}\Delta\tau) + y_{n+2}\right]&\\ n = 0, 2, 4, \cdots&\end{aligned}$$

(You can write a similar relationship for $\mathcal{B}_{n+2}$)

The Time and the Frequency Domains

Transfer Functions

The response of a linear SDOF system to arbitrary loading can be evaluated by a convolution integral in the time domain,

$$x(t) = \int_0^t p(\tau)\, h(t-\tau)\, d\tau,$$

with the unit impulse response function $h(t) = \frac{1}{m\omega_D} \exp(-\zeta\omega_n t)\,\sin(\omega_D t)$, or through the frequency domain using the Fourier integral

$$x(t) = \int_{-\infty}^{+\infty} H(\omega) P(\omega) \exp(i\omega t)\, d\omega,$$

where $H(\omega)$ is the complex frequency response function.

Transfer Functions

These response functions, or *transfer* functions, are connected by the direct and inverse Fourier transforms:

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)\, \exp(-i\omega t)\, dt,$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega)\, \exp(i\omega t)\, d\omega.$$

Relationship of transfer functions

We write the response and its Fourier transform:

$$x(t) = \int_0^t p(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^t p(\tau)h(t-\tau)\,d\tau$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\exp(-i\omega t)\,dt = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^t p(\tau)h(t-\tau)\,d\tau\right]\exp(-i\omega t)\,dt$$

where we changed the lower limit of integration, in the first equation, from 0 to $-\infty$ because $p(\tau) = 0$ for $\tau < 0$.

Relationship of transfer functions

We write the response and its Fourier transform:

$$x(t) = \int_0^t p(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^t p(\tau)h(t-\tau)\,d\tau$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\exp(-i\omega t)\,dt = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^t p(\tau)h(t-\tau)\,d\tau\right]\exp(-i\omega t)\,dt$$

where we changed the lower limit of integration, in the first equation, from 0 to $-\infty$ because $p(\tau) = 0$ for $\tau < 0$.
Since $h(t-\tau) = 0$ for $\tau > t$, the upper limit of the second integral in the second equation can be changed from t to $+\infty$,

$$X(\omega) = \lim_{s\to\infty}\int_{-s}^{+s}\int_{-s}^{+s} p(\tau)h(t-\tau)\exp(-i\omega t)\,dt\,d\tau$$

Relationship of transfer functions

Introducing a new variable $\theta = t - \tau$ we have

$$X(\omega) = \lim_{s\to\infty} \int_{-s}^{+s} p(\tau) \exp(-i\omega\tau)\, d\tau \int_{-s-\tau}^{+s-\tau} h(\theta) \exp(-i\omega\theta)\, d\theta$$

with $\lim\limits_{s\to\infty} s - \tau = \infty$, we finally have

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{+\infty} p(\tau) \exp(-i\omega\tau)\, d\tau \int_{-\infty}^{+\infty} h(\theta) \exp(-i\omega\theta)\, d\theta \\ &= P(\omega) \int_{-\infty}^{+\infty} h(\theta) \exp(-i\omega\theta)\, d\theta = P(\omega)\, H(\omega) \end{aligned}$$

where we have recognized that the first integral is the Fourier transform of $p(t)$.

Relationship of transfer functions

Our last relation was

$$X(\omega) = P(\omega) \int_{-\infty}^{+\infty} h(\theta) \exp(-i\omega\theta)\, d\theta$$

but $X(\omega) = H(\omega)P(\omega)$, so that, noting that in the above equation the last integral is just the Fourier transform of $h(\theta)$, we may conclude that, effectively, $H(\omega)$ and $h(t)$ form a Fourier transform pair.