Multi Degrees of Freedom Systems
MDOF

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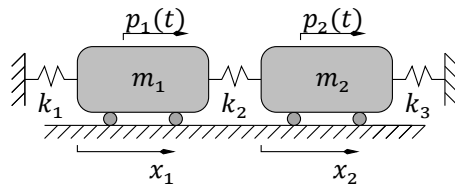 Properties of Structural Matrices
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Consider an undamped system with two masses and two degrees of freedom.

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We can separate the two masses, single out the spring forces and, using the D'Alembert Principle, the inertial forces and, finally. write an equation of dynamic equilibrium for each mass.

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = p_1(t)$$

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = p_2(t)$$

The equation of motion of a 2DOF system

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With some little rearrangement we have a system of two linear differential equations in two variables, $x_1(t)$ and $x_2(t)$:

$$\begin{cases} m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = p_1(t), \\ m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = p_2(t). \end{cases}$$

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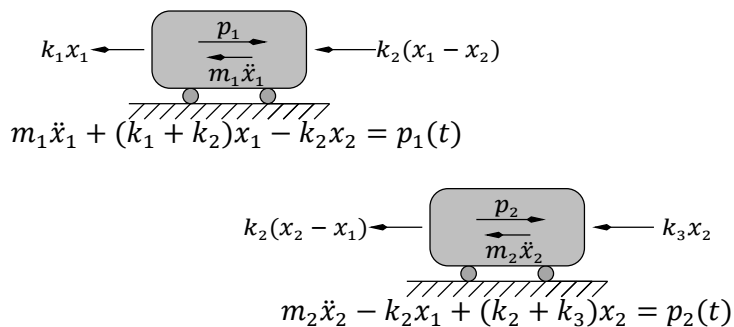Properties of Structural Matrices
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Introducing the loading vector $\mathbf{p}$, the vector of inertial forces $\mathbf{f}_I$ and the vector of elastic forces $\mathbf{f}_S$,

$$\mathbf{p} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}, \quad \mathbf{f}_I = \begin{Bmatrix} f_{I,1} \\ f_{I,2} \end{Bmatrix}, \quad \mathbf{f}_S = \begin{Bmatrix} f_{S,1} \\ f_{S,2} \end{Bmatrix}$$

we can write a vectorial equation of equilibrium:

$$\mathbf{f_I} + \mathbf{f_S} = \mathbf{p}(t).$$

$\mathbf{f}_S = \mathbf{K}\,\mathbf{x}$

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It is possible to write the linear relationship between $\mathbf{f}_S$ and the vector of displacements $\mathbf{x} = \{x_1 x_2\}^T$ in terms of a matrix product, introducing the so called *stiffness matrix* $\mathbf{K}$.
In our example it is

$$\mathbf{f}_S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \mathbf{x} = \mathbf{K}\,\mathbf{x}$$

The stiffness matrix $\mathbf{K}$ has a number of rows equal to the number of elastic forces, i.e., one force for each *DOF* and a number of columns equal to the number of the *DOF*.
The stiffness matrix $\mathbf{K}$ is hence a *square matrix* $\underset{\text{ndof}\times\text{ndof}}{\mathbf{K}}$

$\mathbf{f}_I = \mathbf{M}\,\ddot{\mathbf{x}}$

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Analogously, introducing the *mass matrix* $\mathbf{M}$ that, for our example, is

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

we can write

$$\mathbf{f}_I = \mathbf{M}\,\ddot{\mathbf{x}}.$$

Also the mass matrix $\mathbf{M}$ is a square matrix, with number of rows and columns equal to the number of *DOF*'s.

Matrix Equation

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Finally it is possible to write the equation of motion in matrix format:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}(t).$$

Of course it is possible to take into consideration also the damping forces, taking into account the velocity vector $\dot{\mathbf{x}}$ and introducing a damping matrix $\mathbf{C}$ too, so that we can eventually write

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}(t).$$

But today we are focused on undamped systems...

Properties of K

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- $\mathbf{K}$ is symmetrical.
- $\mathbf{K}$ is a positive definite matrix.

Properties of K: symmetry

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The elastic force exerted on mass i due to an unit displacement of mass j, $f_{S,i} = k_{ij}$ is equal to the force k_{ji} exerted on mass j due to an unit diplacement of mass i, in virtue of *Betti's theorem* (also known as Maxwell-Betti reciprocal work theorem).

The strain energy associated with an imposed displacement vector $\mathbf{x}$ is $V = \frac{1}{2}\,\mathbf{x}^T \cdot \mathbf{f}$ where f is the vector of the elastic forces that cause the displacement, so we can write

$$V = \frac{1}{2}\,\mathbf{x}^T\mathbf{K}\mathbf{x}.$$

Now consider two sets of displacements, $\mathbf{x}_a$ and $\mathbf{x}_b$ and write ① the strain energy associated with first applying $\mathbf{x}_a$ and later $\mathbf{x}_b$:

$$V_{ab} = \frac{1}{2}\,\mathbf{x}_a^T\mathbf{K}\mathbf{x}_a + \frac{1}{2}\,\mathbf{x}_b^T\mathbf{K}\mathbf{x}_b + \mathbf{x}_b^T\mathbf{K}\mathbf{x}_a$$

and ② the strain energy associated with first applying $\mathbf{x}_b$ and later $\mathbf{x}_a$:

$$V_{ba} = \frac{1}{2}\,\mathbf{x}_b^T\mathbf{K}\mathbf{x}_b + \frac{1}{2}\,\mathbf{x}_a^T\mathbf{K}\mathbf{x}_a + \mathbf{x}_a^T\mathbf{K}\mathbf{x}_b.$$

Because $V_{ab} = V_{ba}$ (the final deformed configuration is the same) it is $\mathbf{x}_b^T\mathbf{K}\mathbf{x}_a = \mathbf{x}_a^T\mathbf{K}\mathbf{x}_b$ and, because $\mathbf{x}_b^T\mathbf{K}\mathbf{x}_a = \mathbf{x}_a^T\mathbf{K}^T\mathbf{x}_b$ we can conclude that $\mathbf{K}^T = \mathbf{K}$, i.e., $\mathbf{K}$ is a symmetrical matrix.

Properties of K: definite positivity

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The strain energy V for a discrete system is

$$V = \frac{1}{2}\mathbf{x}^T\mathbf{f}_S,$$

and expressing $\mathbf{f}_S$ in terms of $\mathbf{K}$ and $\mathbf{x}$ we have

$$V = \frac{1}{2}\mathbf{x}^T\mathbf{K}\,\mathbf{x},$$

and because the strain energy is positive for $\mathbf{x} \neq \mathbf{0}$ it follows that $\mathbf{K}$ is definite positive.

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Restricting our discussion to systems whose degrees of freedom are the displacements of a set of discrete masses, we have that the mass matrix is a diagonal matrix, with all its diagonal elements greater than zero. Such a matrix is symmetrical and definite positive.
Both the mass and the stiffness matrix are symmetrical and definite positive.

Note that the kinetic energy for a discrete system can be written

$$T = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\,\dot{\mathbf{x}}.$$

Generalisation of previous results

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The findings in the previous two slides can be generalised to the *structural matrices* of generic structural systems, with two main exceptions.

1. For a general structural system, in which not all DOFs are related to a mass, $\mathbf{M}$ could be *semi-definite* positive, that is for some particular displacement vector the kinetic energy is zero.

2. For a general structural system subjected to axial loads, due to the presence of *geometrical stiffness* it is possible that, for some particular configuration of the axial loads, a displacement vector exists, for which the strain energy is zero and consequently the matrix $\mathbf{K}$ is *semi-definite* positive.

The problem

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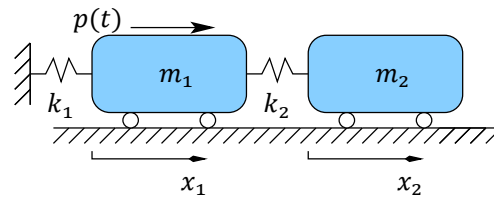Structural Matrices
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Steady–state solution: graphical statement of the problem

$$k_1 = 2k, \quad k_2 = k; \qquad m_1 = 2m, \quad m_2 = 1m;$$
$$p(t) = p_0 \sin \omega t.$$

The equations of motion

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = p_0 \sin \omega t,$$
$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0.$$

... but we prefer the matrix notation ...

The steady state solution

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We prefer the matrix notation because we can find the steady-state response of a *SDOF* system *exactly* as we found the s-s solution for a SDOF system.
Substituting $\mathbf{x}(t) = \boldsymbol{\xi} \sin \omega t$ in the equation of motion and simplifying $\sin \omega t$,

$$k \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \boldsymbol{\xi} - m\omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\xi} = p_0 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

dividing by k, with $\omega_0^2 = k/m$, $\beta^2 = \omega^2/\omega_0^2$ and $\Delta_{st} = p_0/k$ the above equation can be written

$$\left(\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} - \beta^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \boldsymbol{\xi} = \begin{bmatrix} 3 - 2\beta^2 & -1 \\ -1 & 1 - \beta^2 \end{bmatrix} \boldsymbol{\xi} = \Delta_{st} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

The steady state solution

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The determinant of the matrix of coefficients is

$$\text{Det} \left(\begin{bmatrix} 3 - 2\beta^2 & -1 \\ -1 & 1 - \beta^2 \end{bmatrix} \right) = 2\beta^4 - 5\beta^2 + 2$$

but we'll find convenient to write the polynomial in β in terms of its roots

$$\text{Det} = 2 \times (\beta^2 - 1/2) \times (\beta^2 - 2).$$

Solving for $\boldsymbol{\xi}/\Delta_{st}$ in terms of the inverse of the coefficient matrix gives

$$\frac{\boldsymbol{\xi}}{\Delta_{st}} = \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{bmatrix} 1 - \beta^2 & 1 \\ 1 & 3 - 2\beta^2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix} \rightarrow$$

$$\mathbf{x}_{s-s} = \frac{\Delta_{st}}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix} \sin \omega t.$$

The solution, graphically

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steady-state response for a 2 dof system, harmonic load

Comment to the Steady State Solution

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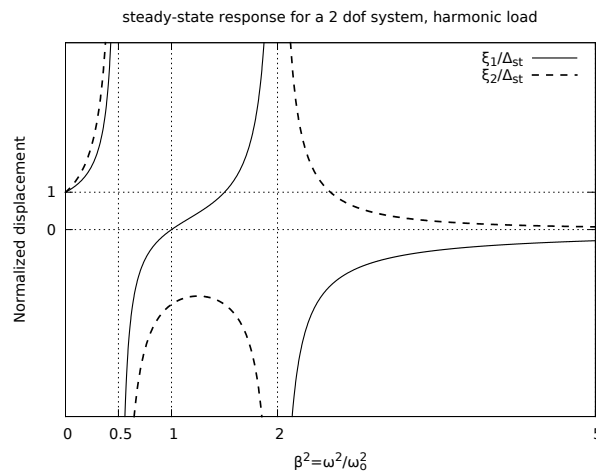Matrices are Linear Operators
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The steady state solution is

$$\mathbf{x}_{\text{s-s}} = \Delta_{\text{st}} \frac{1}{2(\beta^2 - \frac{1}{2})(\beta^2 - 2)} \begin{Bmatrix} 1 - \beta^2 \\ 1 \end{Bmatrix} \sin \omega t.$$

As it's apparent in the previous slide, we have two different values of the excitation frequency for which the *dynamic amplification factor* goes to infinity.

For an undamped SDOF system, we had a single frequency of excitation that excites a *resonant response*, now for a *two* degrees of freedom system we have *two* different excitation frequencies that excite a resonant response.

We know how to compute a particular integral for a MDOF system (at least for a harmonic loading), what do we miss to be able to determine the integral of motion?

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Homogeneous equation of motion

To understand the behaviour of a *MDOF* system, we have to study the homogeneous solution.

Let's start writing the homogeneous equation of motion,

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{0}.$$

The solution, in analogy with the *SDOF* case, can be written in terms of a harmonic function of unknown frequency and, using the concept of separation of variables, of a constant vector, the so called *shape vector* $\boldsymbol{\psi}$:

$$\mathbf{x}(t) = \boldsymbol{\psi}(A\sin\omega t + B\cos\omega t) \qquad \Rightarrow \qquad \ddot{\mathbf{x}}(t) = -\omega^2\,\mathbf{x}(t).$$

Substituting in the equation of motion, we have

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\boldsymbol{\psi}(A\sin\omega t + B\cos\omega t) = \mathbf{0}$$

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Eigenvalues

The previous equation must hold for every value of t, so it can be simplified removing the time dependency:
$$(\mathbf{K} - \omega^2\mathbf{M})\,\boldsymbol{\psi} = \mathbf{0}.$$

The above equation, the **EQUATION OF FREE VIBRATIONS**, is a *set of homogeneous linear equations*, with unknowns ψ_i and whose coefficients depend on the parameter ω^2.

Speaking of homogeneous systems, we know that

- there is always a *trivial solution*, $\boldsymbol{\psi} = \mathbf{0}$, and

- *non-trivial solutions* are possible if the determinant of the matrix of coefficients is equal to zero,
$$\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = 0.$$

The **EIGENVALUES** of the *MDOF* system are the values of ω^2 for which the above equation (the **EQUATION OF FREQUENCIES**) is verified or, in other words, the frequencies of vibration associated with the shapes for which we have equilibrium:

$$\mathbf{K}\boldsymbol{\psi} = \omega^2\mathbf{M}\boldsymbol{\psi} \Leftrightarrow \mathbf{f}_S = \mathbf{f}_I.$$

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Eigenvalues, cont.

For a system with N degrees of freedom the expansion of $\det\left(\mathbf{K} - \omega^2\mathbf{M}\right)$ is an algebraic polynomial of degree N in ω^2 that has N roots, these roots either real or complex conjugate.

In Dynamics of Structures those roots ω_i^2, $i = 1,\dots,N$ are all real because the structural matrices are symmetric matrices.

Moreover, if both $\mathbf{K}$ and $\mathbf{M}$ are positive definite matrices (a condition that you can always enforce for a stable structural system) then all the roots, all the *eigenvalues*, are strictly positive:

$$\omega_i^2 \geq 0, \qquad \text{for } i = 1,\dots,N.$$

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Substituting one of the N roots ω_i^2 in the characteristic equation,

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right) \boldsymbol{\psi}_i = \mathbf{0}$$

the resulting system of $N-1$ linearly independent equations can be solved (except for a scale factor) for $\boldsymbol{\psi}_i$, the eigenvector corresponding to the eigenvalue ω_i^2.

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The scale factor being arbitrary, you have to choose (arbitrarily) the value of one of the components and compute the values of all the other $N-1$ components using the $N-1$ linearly indipendent equations.

It is common to impose to each eigenvector a *normalisation with respect to the mass matrix*, so that

$$\boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i = m$$

where m represents the unit mass.

*Please understand clearly that, substituting **different eigenvalues** in the equation of free vibrations, you have **different linear systems**, leading to **different eigenvectors**.*

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The most general expression (*the general integral*) for the displacement of a homogeneous system is

$$\mathbf{x}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i (A_i \sin \omega_i t + B_i \cos \omega_i t).$$

In the general integral there are $2N$ unknown *constants of integration*, that must be determined in terms of the initial conditions.

Initial Conditions

Usually the initial conditions are expressed in terms of initial displacements and initial velocities $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$, so we start deriving the expression of displacement with respect to time to obtain

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i (A_i \cos \omega_i t - B_i \sin \omega_i t)$$

and evaluating the displacement and velocity for $t = 0$ it is

$$\mathbf{x}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i B_i = \mathbf{x}_0, \qquad \dot{\mathbf{x}}(0) = \sum_{i=1}^{N} \boldsymbol{\psi}_i \omega_i A_i = \dot{\mathbf{x}}_0.$$

The above equations are vector equations, each one corresponding to a system of N equations, so we can compute the $2N$ constants of integration solving the $2N$ equations

$$x_{0,j} = \sum_{i=1}^{N} \psi_{ji} B_i, \qquad \dot{x}_{0,j} = \sum_{i=1}^{N} \psi_{ji} \omega_i A_i =, \qquad j = 1, \dots, N.$$

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Orthogonality - 1

Take into consideration two distinct eigenvalues, ω_r^2 and ω_s^2, and write the characteristic equation for each eigenvalue:

$$\mathbf{K}\boldsymbol{\psi}_r = \omega_r^2 \mathbf{M}\boldsymbol{\psi}_r$$
$$\mathbf{K}\boldsymbol{\psi}_s = \omega_s^2 \mathbf{M}\boldsymbol{\psi}_s$$

premultiply each equation member by the transpose of the *other* eigenvector

$$\boldsymbol{\psi}_s^T \mathbf{K}\boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \mathbf{M}\boldsymbol{\psi}_r$$
$$\boldsymbol{\psi}_r^T \mathbf{K}\boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \mathbf{M}\boldsymbol{\psi}_s$$

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Orthogonality - 2

The term $\boldsymbol{\psi}_s^T \mathbf{K}\boldsymbol{\psi}_r$ is a scalar, hence

$$\boldsymbol{\psi}_s^T \mathbf{K}\boldsymbol{\psi}_r = \left(\boldsymbol{\psi}_s^T \mathbf{K}\boldsymbol{\psi}_r\right)^T = \boldsymbol{\psi}_r^T \mathbf{K}^T \boldsymbol{\psi}_s$$

but $\mathbf{K}$ is symmetrical, $\mathbf{K}^T = \mathbf{K}$ and we have

$$\boldsymbol{\psi}_s^T \mathbf{K}\boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \mathbf{K}\boldsymbol{\psi}_s.$$

By a similar derivation

$$\boldsymbol{\psi}_s^T \mathbf{M}\boldsymbol{\psi}_r = \boldsymbol{\psi}_r^T \mathbf{M}\boldsymbol{\psi}_s.$$

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Substituting our last identities in the previous equations, we have

$$\boldsymbol{\psi}_r^T \mathbf{K} \boldsymbol{\psi}_s = \omega_r^2 \boldsymbol{\psi}_r^T \mathbf{M} \boldsymbol{\psi}_s$$
$$\boldsymbol{\psi}_r^T \mathbf{K} \boldsymbol{\psi}_s = \omega_s^2 \boldsymbol{\psi}_r^T \mathbf{M} \boldsymbol{\psi}_s$$

subtracting member by member we find that

$$(\omega_r^2 - \omega_s^2) \, \boldsymbol{\psi}_r^T \mathbf{M} \boldsymbol{\psi}_s = 0$$

We started with the hypothesis that $\omega_r^2 \neq \omega_s^2$, so for every $r \neq s$ we have that the corresponding eigenvectors are *orthogonal with respect to the mass matrix*

$$\boldsymbol{\psi}_r^T \mathbf{M} \boldsymbol{\psi}_s = 0, \qquad \text{for } r \neq s.$$

Orthogonality - 4

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The eigenvectors are orthogonal also with respect to the stiffness matrix:

$$\boldsymbol{\psi}_s^T \mathbf{K} \boldsymbol{\psi}_r = \omega_r^2 \boldsymbol{\psi}_s^T \mathbf{M} \boldsymbol{\psi}_r = 0, \quad \text{for } r \neq s.$$

By definition

$$M_i = \boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i$$

and consequently

$$\boldsymbol{\psi}_i^T \mathbf{K} \boldsymbol{\psi}_i = \omega_i^2 M_i.$$

M_i is the *modal mass* associated with mode no. i while $K_i \equiv \omega_i^2 M_i$ is the respective *modal stiffness*.

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Eigenvectors are a base

The eigenvectors are reciprocally orthogonal, so they are linearly independent and for every vector $\mathbf{x}$ we can write

$$\mathbf{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j.$$

The coefficients are readily given by premultiplication of $\mathbf{x}$ by $\boldsymbol{\psi}_k^T \mathbf{M}$, because

$$\boldsymbol{\psi}_i^T \mathbf{M} \mathbf{x} = \sum_{j=1}^{N} \boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_j q_j = \boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i q_i = M_i q_i$$

in virtue of the ortogonality of the eigenvectors with respect to the mass matrix, and the above relationship gives

$$q_i = \frac{\boldsymbol{\psi}_i^T \mathbf{M} \mathbf{x}}{M_i}.$$

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Eigenvectors are a base

Generalising our results for the displacement vector to the acceleration vector and expliciting the time dependency, it is

$$\mathbf{x}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j q_j(t), \qquad \ddot{\mathbf{x}}(t) = \sum_{j=1}^{N} \boldsymbol{\psi}_j \ddot{q}_j(t).$$

Introducing $\mathbf{q}(t)$, the *vector of modal coordinates* and $\boldsymbol{\Psi}$, the *eigenvector matrix*, whose columns are the eigenvectors, we can write

$$x_i(t) = \sum_{j=1}^{N} \Psi_{ij} q_j(t), \qquad \ddot{x}_i(t) = \sum_{j=1}^{N} \Psi_{ij} \ddot{q}_j(t),$$

or, in matrix notation

$$\mathbf{x}(t) = \boldsymbol{\Psi}\, \mathbf{q}(t), \qquad \ddot{\mathbf{x}}(t) = \boldsymbol{\Psi}\, \ddot{\mathbf{q}}(t).$$

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EoM in Modal Coordinates...

Substituting the last two equations in the equation of motion,

$$\mathbf{M}\,\boldsymbol{\Psi}\,\ddot{\mathbf{q}} + \mathbf{K}\,\boldsymbol{\Psi}\,\mathbf{q} = \mathbf{p}(t)$$

premultiplying by $\boldsymbol{\Psi}^T$

$$\boldsymbol{\Psi}^T \mathbf{M}\,\boldsymbol{\Psi}\,\ddot{\mathbf{q}} + \boldsymbol{\Psi}^T \mathbf{K}\,\boldsymbol{\Psi}\,\mathbf{q} = \boldsymbol{\Psi}^T \mathbf{p}(t)$$

introducing the so called *starred matrices*, with $\mathbf{p}^\star(t) = \boldsymbol{\Psi}^T \mathbf{p}(t)$, we can finally write

$$\mathbf{M}^\star\,\ddot{\mathbf{q}} + \mathbf{K}^\star\,\mathbf{q} = \mathbf{p}^\star(t).$$

The vector equation above corresponds to the set of scalar equations

$$p_i^\star = \sum m_{ij}^\star \ddot{q}_j + \sum k_{ij}^\star q_j, \qquad i = 1, \dots, N.$$

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... are N independent equations!

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We must examine the structure of the starred symbols.
The generic element, with indexes i and j, of the *starred* matrices can be expressed in terms of single eigenvectors,

$$m_{ij}^\star = \boldsymbol{\psi}_i^T \mathbf{M} \, \boldsymbol{\psi}_j \qquad\qquad = \quad \delta_{ij} M_i,$$
$$k_{ij}^\star = \boldsymbol{\psi}_i^T \mathbf{K} \, \boldsymbol{\psi}_j \qquad\qquad = \omega_i^2 \, \delta_{ij} M_i.$$

where δ_{ij} is the *Kroneker symbol*,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Substituting in the equation of motion, with $p_i^\star = \boldsymbol{\psi}_i^T \mathbf{p}(t)$ we have
a set of uncoupled equations

$$M_i \ddot{q}_i + \omega_i^2 M_i q_i = p_i^\star(t), \qquad i = 1, \dots, N$$

Initial Conditions Revisited

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The initial displacements can be written in modal coordinates,

$$\mathbf{x}_0 = \boldsymbol{\Psi} \, \mathbf{q}_0$$

and premultiplying both members by $\boldsymbol{\Psi}^T \mathbf{M}$ we have the following relationship:

$$\boldsymbol{\Psi}^T \mathbf{M} \, \mathbf{x}_0 = \boldsymbol{\Psi}^T \mathbf{M} \, \boldsymbol{\Psi} \, \mathbf{q}_0 = \mathbf{M}^\star \mathbf{q}_0.$$

Premultiplying by the inverse of $\mathbf{M}^\star$ and taking into account that $\mathbf{M}^\star$ is diagonal,

$$\mathbf{q}_0 = (\mathbf{M}^\star)^{-1} \boldsymbol{\Psi}^T \mathbf{M} \, \mathbf{x}_0 \quad \Rightarrow \quad q_{i0} = \frac{\boldsymbol{\psi}_i^T \mathbf{M} \, \mathbf{x}_0}{M_i}$$

and, analogously,

$$\dot{q}_{i0} = \frac{\boldsymbol{\psi}_i^T \mathbf{M} \, \dot{\mathbf{x}}_0}{M_i}.$$

Note that q_{i0} and $\dot{q}_{i0}$ depend only on the single eigenvector $\boldsymbol{\psi}_i$.

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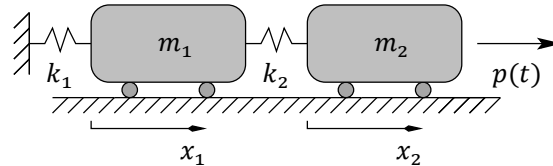
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$$k_1 = 2k, \quad k_2 = 3k; \qquad m_1 = 2m, \quad m_2 = 4m;$$
$$p(t) = p_0 \sin \omega t.$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{p}(t) = \begin{Bmatrix} 0 \\ p_0 \end{Bmatrix} \sin \omega t,$$

$$\mathbf{M} = m \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad \mathbf{K} = k \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}.$$

Equation of frequencies

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The equation of frequencies is

$$\|\mathbf{K} - \omega^2 \mathbf{M}\| = \left\| \begin{matrix} 5k - 2\omega^2 m & -3k \\ -3k & 3k - 4\omega^2 m \end{matrix} \right\| = 0.$$

Developing the determinant

$$(8m^2)\,\omega^4 - (26mk)\,\omega^2 + (6k^2)\,\omega^0 = 0$$

Solving the algebraic equation in ω^2

$$\omega_1^2 = \frac{k}{m}\frac{13 - \sqrt{121}}{8}, \qquad\qquad \omega_2^2 = \frac{k}{m}\frac{13 + \sqrt{121}}{8};$$
$$\omega_1^2 = \frac{1}{4}\frac{k}{m}, \qquad\qquad\qquad \omega_2^2 = 3\frac{k}{m}.$$

Eigenvectors

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Substituting ω_1^2 for ω^2 in the first of the characteristic equations gives the ratio between the components of the first eigenvector,

$$k\,(5 - 2\cdot\frac{1}{4})\psi_{11} = 3k\psi_{21}$$

while substituting ω_2^2 gives

$$k\,(3 - 4\cdot 3)\psi_{12} = 3k\psi_{22}.$$

Solving with the arbitrary assignment $\psi_{11} = \psi_{22} = 1$ gives the *unnormalized* eigenvectors,

$$\boldsymbol{\psi}_1 = \begin{Bmatrix} +1 \\ +\frac{3}{2} \end{Bmatrix}, \quad \boldsymbol{\psi}_2 = \begin{Bmatrix} -3 \\ +1 \end{Bmatrix}.$$

Normalization

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We compute first M_1 and M_2,

$$M_1 = \boldsymbol{\psi}_1^T \mathbf{M}\, \boldsymbol{\psi}_1$$

$$= m \left\{ 1, \quad \tfrac{3}{2} \right\} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \left\{ \begin{matrix} 1 \\ \tfrac{3}{2} \end{matrix} \right\}$$

$$= m \left\{ 2, \quad 6 \right\} \left\{ \begin{matrix} 1 \\ \tfrac{3}{2} \end{matrix} \right\} = 11\, m$$

and, in a similar way, we have $M_2 = 22\, m$; the *adimensional* normalisation factors are

$$\alpha_1 = \sqrt{11} = 3.317, \qquad \alpha_2 = \sqrt{22} = 4.690.$$

Applying the normalisation factors to the respective unnormalised eigenvectors and collecting them in a matrix, we have the *matrix of normalized eigenvectors*

$$\boldsymbol{\Psi} = \begin{bmatrix} +0.30151 & -0.63960 \\ +0.45227 & +0.21320 \end{bmatrix}$$

Modal Loadings

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The modal loading is

$$\mathbf{p}^\star(t) = \boldsymbol{\Psi}^T\, \mathbf{p}(t)$$

$$= p_0 \begin{bmatrix} 1 & 3/2 \\ -3 & 1 \end{bmatrix} \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \sin \omega t$$

$$= p_0 \left\{ \begin{matrix} 3/2 \\ 1 \end{matrix} \right\} \sin \omega t$$

Modal EoM

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Substituting its modal expansion for $\mathbf{x}$ into the equation of motion and premultiplying by $\boldsymbol{\Psi}^T$ we have the uncoupled modal equation of motion

$$\begin{cases} 11\, m\, \ddot{q}_1 + \dfrac{1}{4}\, 11\, m\, \dfrac{k}{m}\, q_1 = \dfrac{3}{2} p_0 \sin \omega t \\[2mm] 22\, m\, \ddot{q}_2 + 3\, 22\, m\, \dfrac{k}{m}\, q_2 = p_0 \sin \omega t \end{cases}$$

Note that all the terms are dimensionally correct. Dividing by M_i both equations, we have

$$\begin{cases} \ddot{q}_1 + \dfrac{1}{4}\, \omega_0^2 q_1 = {}^3\!/\!2\, \dfrac{p_0}{11m} \sin \omega t \\[2mm] \ddot{q}_2 + 3\, \omega_0^2 q_2 = \dfrac{p_0}{22m} \sin \omega t \end{cases}$$

Particular Integral

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We set

$$\xi_1 = C_1 \sin \omega t, \quad \ddot{\xi} = -\omega^2 C_1 \sin \omega t$$

and substitute in the first modal EoM:

$$C_1 (\omega_1^2 - \omega^2) \sin \omega t = \frac{3}{22} \frac{p_0}{k} \frac{k}{m} \sin \omega t$$

solving for C_1

$$C_1 = \frac{3}{22} \Delta \frac{\omega_0^2}{\omega_1^2 - \omega^2}$$

and, analogously,

$$C_2 = \frac{1}{22} \Delta \frac{\omega_0^2}{\omega_2^2 - \omega^2}$$

with $\Delta = p_0/k$.

Integrals

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The integrals, for our loading, are thus

$$\begin{cases} q_1(t) = A_1 \sin \omega_1 t + B_1 \cos \omega_1 t + C_1 \sin \omega t, \\ q_2(t) = A_2 \sin \omega_2 t + B_2 \cos \omega_2 t + C_2 \sin \omega t, \end{cases}$$

and, for a system initially at rest, it is

$$\begin{cases} q_1(t) = C_1 \left(\sin \omega t - \beta_1 \sin \omega_1 t \right), \\ q_2(t) = C_2 \left(\sin \omega t - \beta_2 \sin \omega_2 t \right), \end{cases}$$

where $\beta_i = \omega/\omega_i$
We are interested in structural degrees of freedom, too...

$$\begin{cases} x_1(t) = \left(\psi_{11} q_1(t) + \psi_{12} q_2(t) \right) \\ x_2(t) = \left(\psi_{21} q_1(t) + \psi_{22} q_2(t) \right) \end{cases}$$

The response in modal coordinates

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To have a feeling of the response in modal coordinates, let's say that the frequency of the load is $\omega = 2\omega_0$, hence $\beta_1 = \frac{2.0}{\sqrt{1/4}} = 4$ and $\beta_2 = \frac{2.0}{\sqrt{3}} = 1.15470$.

In the graph above, the responses are plotted against an adimensional time coordinate α with $\alpha = \omega_0 t$, while the ordinates are adimensionalised with respect to $\Delta_{st} = \frac{p_0}{k}$

The response in structural coordinates

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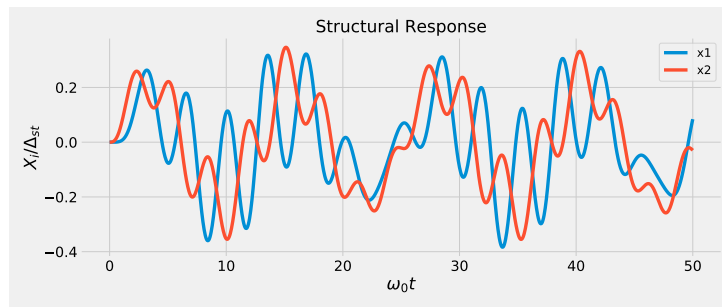
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Using the same normalisation factors, here are the response functions in terms of
$x_1 = \psi_{11}q_1 + \psi_{12}q_2$ and $x_2 = \psi_{21}q_1 + \psi_{22}q_2$:

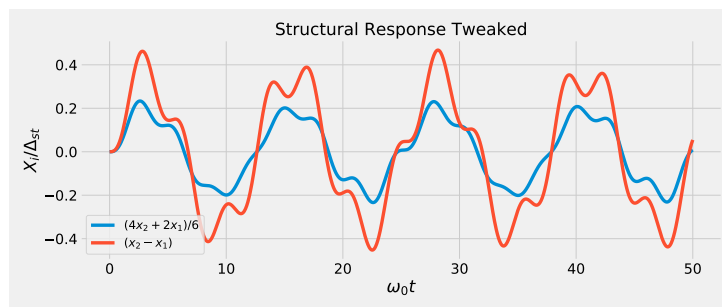# The response in structural coordinates

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And the displacement of the centre of mass plotted along with the difference in displacements.