

Continuous Systems

an example

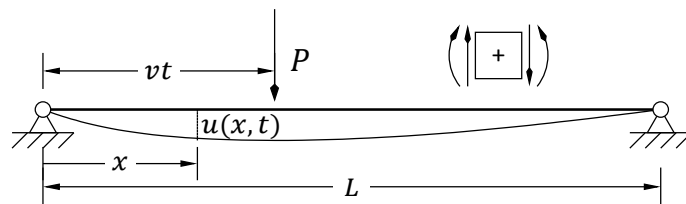
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April 16, 2020

Problem statement



A uniform beam, (unit mass m , flexural stiffness EJ and length L) is loaded by a load P , moving with constant velocity $v(t) = v$ in the time interval $0 \leq t \leq t_0 = L/v = t_0$.

Plot the response in the interval $0 \leq t \leq t_0 = L/v$ in terms of $u(L/2, t)$ and $M_b(L/2, t)$.

NB: the beam is at rest for $t = 0$.

Equation of motion

For a uniform beam, the equation of dynamic equilibrium is

$$m \frac{\partial^2 u(x, t)}{\partial t^2} + EJ \frac{\partial^4 u(x, t)}{\partial x^4} = p(x, t).$$

In our example, the loading function must be defined in terms of $\delta(x)$, the Dirac's delta distribution,

$$p(x, t) = P \delta(x - vt).$$

The Dirac's delta (or distribution) is defined by

$$\delta(x - x_0) \equiv 0 \quad \text{and} \quad \int f(x) \delta(x - x_0) dx = f(x_0).$$

Equation of motion

The solution will be computed by separation of variables

$$u(x, t) = q(t)\phi(x)$$

and modal analysis,

$$u(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

The relevant quantities for the modal analysis, obtained solving the eigenvalue problem that arises from the beam boundary conditions are

$$\begin{aligned} \phi_n(x) &= \sin \beta_n x, & \beta_n &= \frac{n\pi}{L}, \\ m_n &= \frac{mL}{2}, & \omega_n^2 &= \beta_n^4 \frac{EJ}{m} = n^4 \pi^4 \frac{EJ}{mL^4}. \end{aligned}$$

Orthogonality relationships

For an uniform beam, the orthogonality relationships are

$$\begin{aligned} m \int_0^L \phi_n(x)\phi_m(x) dx &= m_n \delta_{nm}, \\ EJ \int_0^L \phi_n(x)\phi_m^{IV}(x) dx &= k_n \delta_{nm} = m_n \omega_n^2 \delta_{nm}. \end{aligned}$$

(the Kronecker's δ_{nm} is a completely different thing from Dirac's δ , OK?).

Decoupling the EOM

Using the orthogonality relationships, we can write an infinity of uncoupled equation of motion for the modal coordinates.

- 1 The equation of motion is written in terms of the series representation of $u(x, t)$:

$$m \sum_{m=1}^{\infty} \ddot{q}_m \phi_m + EJ \sum_{m=1}^{\infty} q_m \phi_m^{IV} = P \delta(x - vt),$$

- 2 every term is multiplied by ϕ_n and integrated over the length of the beam

$$m \int_0^L \phi_n \sum_{m=1}^{\infty} \ddot{q}_m \phi_m dx + EJ \int_0^L \phi_n \sum_{m=1}^{\infty} q_m \phi_m^{IV} dx = P \int_0^L \phi_n \delta(x - vt) dx, \quad n = 1, \dots, \infty$$

- 3 we use the orthogonality relationships and the definition of δ ,

$$m_n \ddot{q}(t) + k_n q(t) = P \phi_n(vt) = P \sin \frac{n\pi vt}{L}, \quad n = 1, \dots, \infty.$$

Solutions

Considering that

- the initial conditions are zero for all the modal equations,
- for each mode we have a *different* excitation frequency $\bar{\omega}_n = n\pi v/L$ (and also $\beta_n = \bar{\omega}_n/\omega_n$),

the individual solutions are given by

$$q_n(t) = \frac{P}{k_n} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}$$

and, with $k_n = m_n \omega_n^2 = \frac{mL}{2} n^4 \pi^4 \frac{EJ}{mL^4} = n^4 \pi^4 \frac{EJ}{2L^3}$, it is

$$q_n(t) = \frac{2}{n^4 \pi^4} \frac{PL^3}{EJ} \frac{1}{1 - \beta_n^2} (\sin \bar{\omega}_n t - \beta_n \sin \omega_n t), \quad 0 \leq t \leq \frac{L}{v}.$$

It is apparent that we have *resonance* for $\beta_n = 1$.

Critical Velocity

Let's start from $\beta_1 = \pi v/L/\omega_1 = 1$ and solve for the velocity, say v_1

$$v_1 = \omega_1 L/\pi.$$

It is apparent that v_1 is a critical velocity $v_c = v_1 = \omega_1 L/\pi$ that gives a resonance condition for the first mode response, while for $v = 2 v_c$ the second mode is in resonance, etc.

With the position $v = \kappa v_1$ it is

$$\bar{\omega}_n = \kappa n \omega_1 \quad \text{and} \quad \beta_n = n \kappa \omega_1 / n^2 \omega_1 = \kappa/n$$

and we can rewrite the solution as

$$q_n(t) = \frac{2PL^3}{\pi^4 EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin\left(\frac{\kappa}{n} \omega_n t\right) - \frac{\kappa}{n} \sin \omega_n t \right), \quad 0 \leq t \leq \frac{L}{v}.$$

Adimensional Time Coordinate

Introducing an adimensional time coordinate ξ with $t = t_0 \xi$, noting that $\omega_n = n^2 \omega_1$ we can write

$$\frac{\kappa}{n} \omega_n t = \frac{\kappa}{n} n^2 \omega_1 \xi t_0 = \kappa n \left(\frac{v_c \pi}{L} \right) \xi \frac{L}{\kappa v_c} = n \pi \xi,$$

substituting in the solution for mode n we have

$$q_n(\xi) = \frac{2}{\pi^4} \frac{PL^3}{EJ} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa} \pi\xi\right) \right), \quad 0 \leq \xi \leq 1.$$

Adimensional Time and Adimensional Position

If we denote with $\mathbb{X}(t)$ the position of the load at time t , it is $\mathbb{X}(t) = vt = \xi L$, or $\xi = \mathbb{X}/L$ and the expression $u(x, \xi) = \sum q_n(\xi)\phi_n(x)$ can be interpreted as the displacement in x when the load is positioned in ξL .

Displacement and Bending Moment

The displacement and the bending moment are given by

$$u(x, \xi) = \frac{2PL^3}{\pi^4 EJ} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\pi\frac{x}{L}\right),$$

$$\begin{aligned} M_b(x, \xi) &= -EJ \frac{\partial^2 u(x, \xi)}{\partial x^2} \\ &= \frac{2PL}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\pi\frac{x}{L}\right). \end{aligned}$$

Normalized Midspan Deflection

If we consider the midspan deflection (bending moment) due to a static load P on the beam, the maximum deflection (bending moment) is expected when the load is placed at midspan, and it is

$$u_{\text{stat}}(L/2, 1/2) = \frac{PL^3}{48EJ} \quad \text{and} \quad M_{b \text{ stat}}(L/2, 1/2) = \frac{PL}{4}.$$

Normalizing the midspan displacement with respect to the maximum static displacement, we write

$$\Delta(\xi) = \frac{u}{u_{\text{stat}}} = \frac{96}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right).$$

Eventually we introduce a notation for the partial sum of the first N terms:

$$\Delta_N(\xi) = \frac{96}{\pi^4} \sum_{n=1}^N \frac{1}{n^2(n^2 - \kappa^2)} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right).$$

Normalized Midspan Bending Moment

Analogously, normalizing with respect to the maximum static bending moment, it is

$$\mu(\xi) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right),$$

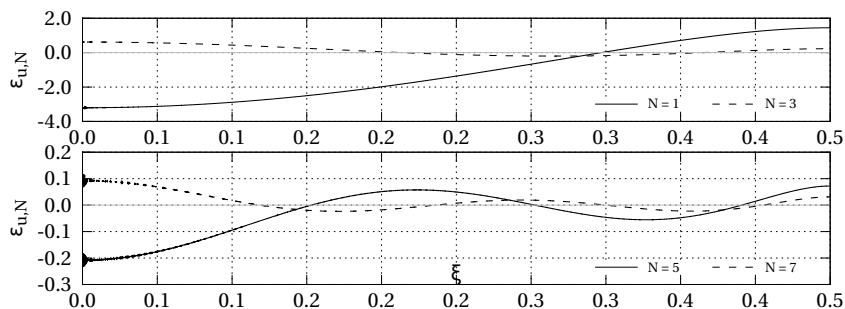
the partial sum being denoted by

$$\mu_N(\xi) = \frac{8}{\pi^2} \sum_{n=1}^N \frac{1}{n^2 - \kappa^2} \left(\sin(n\pi\xi) - \frac{\kappa}{n} \sin\left(\frac{n^2}{\kappa}\pi\xi\right) \right) \sin\left(n\frac{\pi}{2}\right).$$

Error Estimates

To appreciate the approximation inherent in a truncated series, we compare the truncated series computed for $\kappa = 10^{-6}$ with the static response $\Delta_{\text{stat}}(\xi) = 3\xi - 4\xi^3$ introducing a percent error function

$$\epsilon_{u,N}(\xi) = 100 \left(1 - \frac{\Delta_N(\xi)|_{\kappa=10^{-6}}}{\Delta_{\text{stat}}(\xi)} \right) \quad \text{for } 0 \leq \xi \leq 1/2,$$

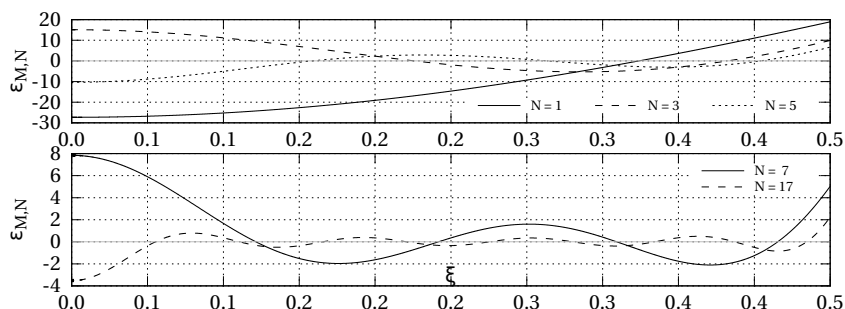


Using 4 terms ($N = 7$) the absolute error is not greater than $1/1000$.

Error Estimates

Analogously we can use the midspan bending moment, normalized with respect to $PL/4$, $\mu_{\text{stat}}(\xi) = 2\xi$ to define another percent error function

$$\epsilon_{M,N} = 100 \left(1 - \frac{\mu_N(\xi)|_{\kappa=10^{-6}}}{\mu_{\text{stat}}(\xi)} \right)$$

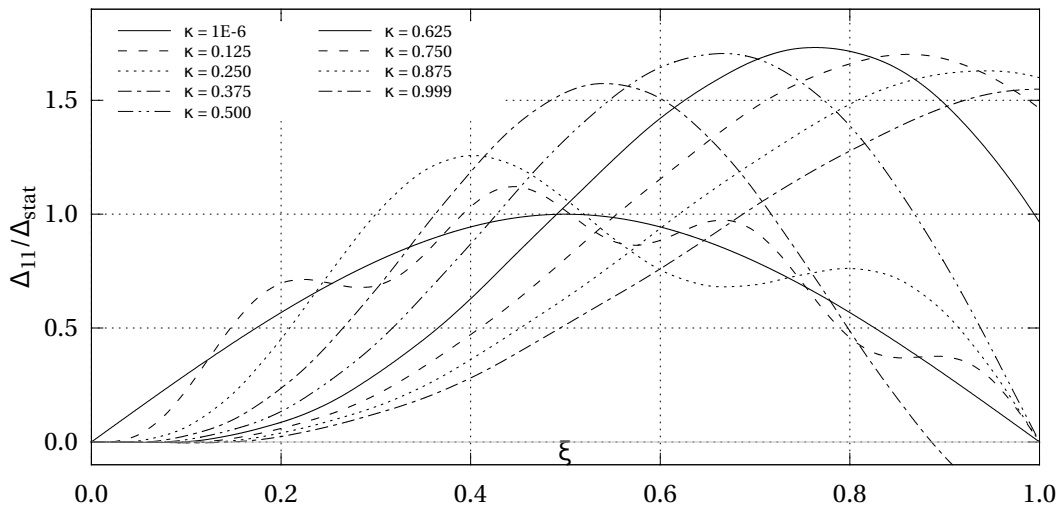


With 8 terms ($N = 17$) terms in the series, still the absolute error is greater than 3%.

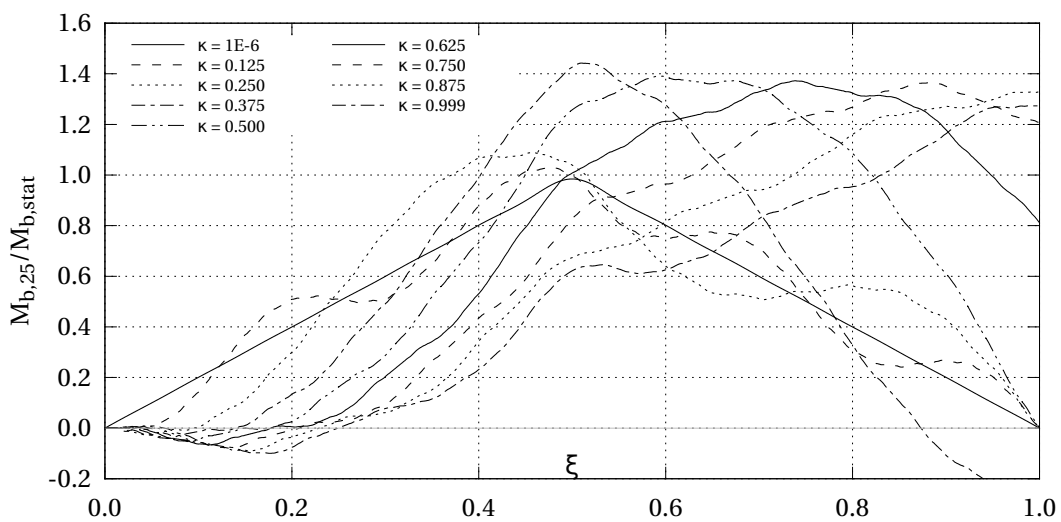
The Plots

Eventually, we plot the normalized displacement and the normalized bending moment for different values of κ , i.e., for different velocities.

For the displacement I used $N = 11$ while for the bending moment I used $N = 25$.



Normalized Midspan Displacement.
(for different velocities $v = \kappa v_c$)



Normalized Midspan Bending Moment.
(for different velocities $v = \kappa v_c$)